Torsion points on GL_2-type abelian varieties

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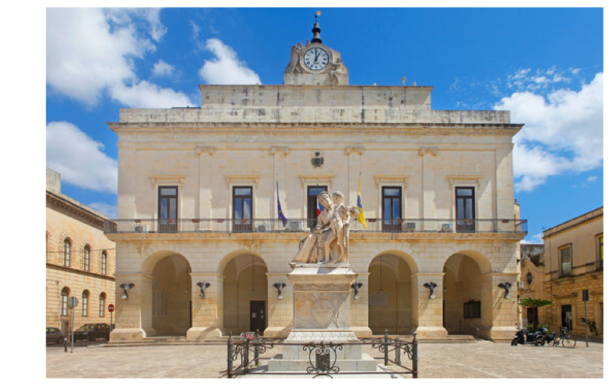

Motivation

Let $E/\mathbb{Q}$ be an elliptic curve. By **Mazur's Theorem**, $\mathrm{Tors}(E)(\mathbb{Q})$ is one of

$$\{1\}, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{12}, C_2 \times C_2, C_2 \times C_4, C_2 \times C_6, C_2 \times C_8$$

Partial generalisations:

- If E/K is an elliptic curve, then $|\mathrm{Tors}(E)(K)| \leq (3^{[K:\mathbb{Q}]+1})^2$ [1].
- If $A/\mathbb{Q}$ is an abelian surface with potential quaternionic multiplication, then $|\mathrm{Tors}(A)(\mathbb{Q})| \leq 18$ [2].
- If $A/\mathbb{Q}$ is an abelian surface with potential quaternionic multiplication **of** GL_2**-type**, then $\mathrm{Tors}(A)(\mathbb{Q})$ is one of
$$\{1\}, C_2, C_3, C_2^2, C_3^2.$$

Lang's question

Given A/K an abelian variety over a number field, if for all but finitely many finite places $\mathfrak{p}$

$$|\tilde{A}(\mathbb{F}_\mathfrak{p})| \equiv 0 \pmod{m},$$

is it true that (up to isogeny)

$$|\mathrm{Tors}(A)(K)| \equiv 0 \pmod{m}?$$

Galois representations and Katz

Let $\rho_{A,\ell} : G_K \to \mathrm{GL}(V_\ell A)$ be the ℓ-adic Galois representation attached to A. Then

$$|\tilde{A}(\mathbb{F}_\mathfrak{p})| = \det(1 - \rho_{A,\ell}(\mathrm{Frob}_\mathfrak{p}))$$

for every $\mathfrak{p}$ of good reduction, and there exists A' isogenous to A with

$$|\mathrm{Tors}(A')(K)| \equiv 0 \pmod{\ell^n}$$

if and only if there are two lattices $T' \subseteq T$, with $T = T_\ell A, T' = T_\ell A'$ such that G_K acts trivially on T/T' and $|T/T'| = \ell^n$.

This is true for elliptic curves[3]!

GL_2-type abelian varieties

An abelian variety A/K is **of** GL_2**-type** if there exists an embedding $F \hookrightarrow \mathrm{End}_K(A) \otimes_\mathbb{Z} \mathbb{Q}$ with F a number field of degree $g = \dim(A)$. In this case, there is a system of **strictly compatible F-rational 2-dimensional Galois representation**

$$\rho_{A,\lambda} : G_K \to \mathrm{GL}_2(F_\lambda)$$

for each prime λ over each rational prime ℓ, and

$$\det(1 - \rho_{A,\ell}(\mathrm{Frob}_\mathfrak{p})) = N_{F/\mathbb{Q}}(\det(1 - \rho_{A,\lambda}(\mathrm{Frob}_\mathfrak{p})))$$

for every $\mathfrak{p}$ of good reduction.

Main theorem [4]

Let A/K be an abelian variety of GL_2-type over a number field.

If for all but finitely many finite places $\mathfrak{p}$ of K one has

$$\det(1 - \rho_{A,\lambda}(\mathrm{Frob}_\mathfrak{p})) \equiv 0 \pmod{\lambda^n},$$

then (up to isogeny)

$$|\mathrm{Tors}(A)(K)| \equiv 0 \pmod{\ell^{n\mathfrak{f}_\lambda}}.$$

In particular, if $|\tilde{A}(\mathbb{F}_\mathfrak{p})| \equiv 0 \pmod{\ell}$ for every prime of good reduction, then (up to isogeny)

$$|\mathrm{Tors}(A)(K)| \equiv 0 \pmod{\ell}.$$

Conjecture (La Congettura Di Maglie)

Let $A/\mathbb{Q}$ be a **simple non-CM g-dimensional abelian variety of** GL_2**-type**. The possible orders of $\mathrm{Tors}(A)(\mathbb{Q})$ are listed below.

g	$\lvert\mathrm{Tors}(A)(\mathbb{Q})\rvert$
2	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 28, 31, 37, 44, 56
3	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 28, 29, 31, 32, 36, 37, 38, 40, 44, 46, 49, 58, 80, 83, 92, 160
4	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 34, 35, 36, 37, 38, 40, 41, 43, 44, 48, 49, 52, 55, 56, 57, 61, 63, 64, 68, 71, 72, 73, 74, 76, 80, 82, 88, 136, 146, 152, 155, 176
5	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 34, 35, 36, 38, 40, 41, 43, 44, 46, 48, 50, 52, 53, 55, 56, 63, 64, 68, 72, 80, 92, 96, 106, 110, 144, 160, 192, 288, 320

Let $A/\mathbb{Q}$ be a **simple non-CM g-dimensional abelian variety of** GL_2**-type** and let ℓ be a prime such that A admits a ℓ-torsion point. The possible values for ℓ are listed below.

g	ℓ
2	2, 3, 5, 7, 11, 13, 17, 19, 23, 31, 37
3	2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 83
4	2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 61, 71, 73
5	2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 43, 53

The idea

The Jacobian of the modular curve of level $\Gamma_1(N)$ can (up to isogeny) be written as a product of irreducible abelian varieties A_f, corresponding to **Hecke eigenforms** f of weight 2, level N and Nebentypus χ, for some χ.

Simple abelian varieties of GL_2-type over $\mathbb{Q}$ are **modular**, hence isogenous to A_f for some $f(q) = \Sigma_{n \geq 1} a_n(f) q^n$, with $F = \mathbb{Q}(\{a_n(f)\})$ being a degree $\dim(A)$ number field.

Then $\rho_{A,\lambda} = \rho_f$, and

$$\det(1 - t\rho_{A,\lambda}(\mathrm{Frob}_p)) = 1 - a_p(f)t + \chi(p)pt^2.$$

The algorithm [5]

- Loop over all (non-CM) Hecke newforms f of weight 2 and level up to a fixed bound, having coefficient field of degree g (we accessed LMFDB).
- For each, compute $1 - a_p(f) + \chi(p)p$, and factorize its norm over $\mathbb{Q}$.
- For each ℓ in the factorisation compute $\sup_{\lambda|\ell} \ell^{\mathfrak{f}_\lambda n}$, where $1 - a_p(f) + \chi(p)p \equiv 0 \pmod{\lambda^n}$.
- Obtain the predicted torsion order, as $\Pi_\ell \sup_{\lambda|\ell} \ell^{\mathfrak{f}_\lambda n}$

Open problems

Prove the conjectures! Ideas on how are welcome!

Something new!

One abelian variety in the isogeny class of the simple abelian surface associated to the newform with `LMFDB` [6] label `2190.2.a.v` has torsion order 37, a result previously unmatched! (see `https://www.lmfdb.org/Genus2Curve/Q/?torsion_order=37`)

References

[1] Joseph Oesterlé.
Torsion des courbes elliptiques sur les corps de nombres, 1994.
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[2] Jef Laga, Ciaran Schembri, Ari Shnidman, and John Voight.
Rational torsion points on abelian surfaces with quaternionic multiplication.
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[3] Nicholas M. Katz.
Galois properties of torsion points on abelian varieties.
Inventiones Mathematicae, 62:481–502, 1981.

[4] Jessica Alessandrì and Nirvana Coppola.
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arXiv:2602.21047, 2026.

[5] Jessica Alessandrì and Nirvana Coppola.
Magma code for computing torsion orders of GL_2-type abelian varieties, 2026.
https://github.com/NirvanaC93/Torsion-of-GL2-abelian-varieties.

[6] LMFDB - The L-functions and modular forms database.
https://www.lmfdb.org/.

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