A p-adic cohomological approach to congruences of meromorphic modular forms

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The motivating congruences

Zhang's numerical congruences [Zha25]. For an elliptic curve $C/\mathbb{Q}$ with good reduction at p and $a_p(C) = p + 1 - |C(\mathbb{F}_p)|$, Zhang observed:

$$a_p\left(\frac{E_4}{j - j(C)}\right) \equiv a_p(C)^2 \pmod{p}$$

and systematic congruences linking **Fourier coefficients** of meromorphic modular forms to **Frobenius eigenvalues of the pole** elliptic curves. For $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$ and $q = e^{2\pi i \tau}$, we have

$E_4(q) := 1 + 240 \sum_{n \geq 1} \left(\sum_{d|n} d^3\right) q^n$ the weight four **Eisenstein series**,

$j(\tau)$ the j-function associated to the elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

Goal. Give a uniform **p-adic cohomological** interpretation of these congruences.

Key tool: Rigid residue sequence and log-crystalline cohomology to produce a Frobenius-stable $\mathbb{Z}_p$-lattice in the rigid cohomology.

Geometric setup

Consider X a fine moduli scheme of generalized elliptic curves with extra level structure admitting a **smooth model** over $\mathbb{Z}_p$. Let $\pi : \mathcal{E} \to X$ be the universal elliptic curve and C be the cuspidal subgroup. Consider the following coherent sheaves

1. $\omega := \pi_* \Omega^1_{\mathcal{E}/X}(\log \pi^{-1}(C))$ the *Hodge line bundle* from the universal object
2. $(\mathcal{H}, \nabla) := R^1\pi_*(\Omega^\bullet_{\mathcal{E}/X})$ the rank-2 vector bundle from *relative de Rham cohomology*, with *Gauss-Manin connection* fitting into exact sequence
$$0 \to \omega \to \mathcal{H} \to \omega^{-1} \to 0,$$
3. $\mathcal{H}_k = \mathrm{Sym}^k \mathcal{H}$ with the induced Hodge filtration:
$$\mathcal{H}_k = \mathrm{Fil}^0 \mathcal{H}_k \supset \cdots \supset \mathrm{Fil}^{k+1}\mathcal{H}_k = 0, \quad \mathrm{Gr}^i \mathcal{H}_k \cong \omega^{k-2i}$$

Modular forms:

$$M_k(X) = \Gamma(X, \omega^k), \quad S_k(X) = \Gamma(X, \omega^{k-2} \otimes \Omega^1_X)$$

Eichler–Shimura exact sequence:

$$0 \to M_{k+2}(X) \to H^1_{\mathrm{dR}}(X, \mathcal{H}_k) \to S_{k+2}(X)^\vee \to 0$$

Serre derivative

$$0 \to \omega_{-k} \xrightarrow{\theta^{k+1}} \omega_{k+2} \to \mathcal{H}_k \otimes_X \Omega^1_X(\log C)/\nabla_k(\mathcal{H}_k) \to 0$$

where $\theta = q\frac{d}{dq}$ on q-expansion.

The residue sequence with hidden structure

Fix an effective divisor $S = \sum_i \alpha_i$ on $X \setminus C$, with $\alpha_i \in X(\mathbb{Z}_p)$ having distinct reductions mod p. We have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{\mathrm{dR}}(X^{\mathrm{rig}}, \mathcal{H}_k^{\mathrm{rig}}) & \longrightarrow & H^1_{\mathrm{dR}}(X \setminus S, \mathcal{H}_k^{\mathrm{rig}}) & \xrightarrow{\mathrm{Res}} & \bigoplus_{\alpha \in S} \mathrm{Sym}^k H^1_{\mathrm{dR}}(\mathcal{E}_\alpha)[1] & \longrightarrow & 0 \\ & & & & \downarrow{\wr} & & & & \\ & & & & M_{k+2}^{\mathrm{mero},S}/\theta^{k+1} M_{-k}^{\mathrm{mero},S} & & & & \end{array}$$

The residue sequence is exact in the category of **filtered-φ modules** [CI10].
The comparison result of Baldassarri-Chiarellotto [BC94] allows to remove small disks at the poles in the rigid category

$$H^1_{\mathrm{dR}}(X \setminus S, \mathcal{H}_k) \cong H^1_{\mathrm{dR}}(V_\lambda, \mathcal{H}_k^{\mathrm{rig}}) \cong M_{k+2}^{\dagger,\lambda}/\theta^{k+1} M_{-k}^{\dagger,\lambda}.$$

An overconvergent idea

To explicitly extract information about the Frobenius structure we apply Katz-Lubin theory of **canonical subgroup** [Kat73].

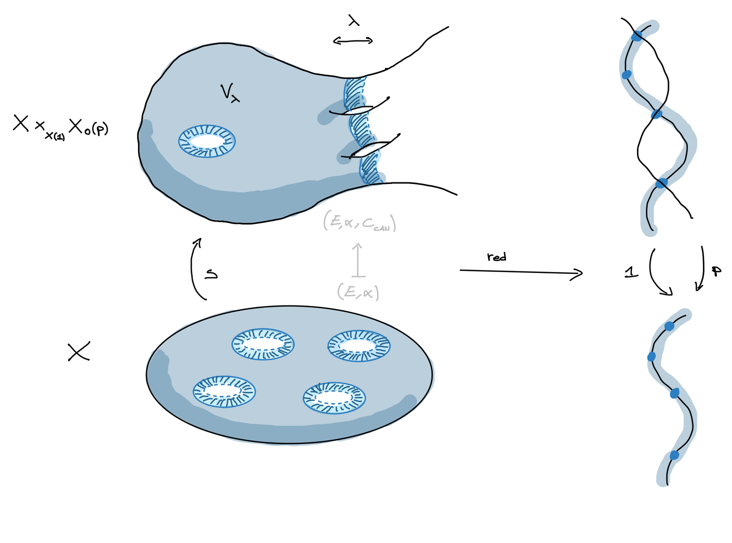

Figure 1. Canonical lift

These spaces are naturally endowed with two modular operators V_p, U_p acting on q-expansions $f(q) = \sum_{n \geq 0} a_n(f) q^n$ as

$$V_p f(q) = \sum_{n \geq 0} a_n(f) q^{np}, \quad U_p f(q) = \sum_{n \geq 0} a_{np}(f) q^n$$

and a normalized eigenform for U_p has eigenvalue $a_p(f)$. These operators induce endomorphisms $\mathrm{Frob}, \mathrm{Ver}$ on $H^1_{\mathrm{dR}}(V_\lambda, \mathcal{H}_k^{\mathrm{rig}})$ [Col96] such that

$$\boxed{\mathrm{Frob} \circ \mathrm{Ver} = \mathrm{Ver} \circ \mathrm{Frob} = p^{k+1} \quad \text{on } H^1_{\mathrm{dR}}(V_\lambda, \mathcal{H}_k^{\mathrm{rig}}).}$$

Theorem 1 (B. 2026)

Let $P(X)$ (resp. $Q(X)$) be the characteristic polynomial of Frobenius on $\mathrm{Sym}^k H^1_{\mathrm{dR}}(\mathcal{E}_\alpha)[1]$ (resp. $H^1_{\mathrm{dR}}(X^{\mathrm{rig}}, \mathcal{H}_k^{\mathrm{rig}})$), and set $P(X)Q(X) = \sum_{i=0}^M e_i X^i$ with $M = \dim H^1_{\mathrm{dR}}(X^{\mathrm{rig}}, \mathcal{H}_k^{\mathrm{rig}}) + k + 1$.

For every $f \in M_{k+2}^{\mathrm{mero},\{\alpha\}}$:

$$\boxed{\sum_{i=0}^M e_{M-i}\, p^{M-i}\, U_p^i(f) \in \theta^{k+1}\left(M_{-k,\lambda}^{\dagger,\{\alpha\}}\right).}$$

Proof idea: Annihilate $H^1_{\mathrm{dR}}(V_\lambda, \mathcal{H}_k^{\mathrm{rig}})$ via $P \cdot Q(\mathrm{Frob})$, enlarge S to supersingular locus to make the Frobenius explicit, then substitute $\mathrm{Ver}^M \circ \mathrm{Frob}^i \leftrightarrow p^{M-i} U_p^i$.

Frobenius lattice and q-expansions

Consider the infinitesimal neighborhood $\mathrm{Spf}(\mathbb{Z}_p[[q^{1/N}]])$ of the cusp $[\infty]$ of the formal model $\mathfrak{X}$. The fiber product with the universal elliptic curve gives rise to the **Tate curve**

$$\begin{array}{ccc} \mathrm{Tate}(q^{1/N}) & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathrm{Spf}(\mathbb{Z}_p[[q^{1/N}]]) & \xrightarrow{\iota} & \mathfrak{X}. \end{array}$$

Thanks to the formal smooth lift of the variety we can pull back the *log-crystalline cohomology* at infinity to obtain integral q-expansion description

$$H^1_{\log-\mathrm{crys}}((\overline{X}, \overline{S})/\mathbb{Z}_p, \mathcal{H}_k) \to \mathbb{Z}_p[[q^{1/N}]]/\theta^{k+1}(\mathbb{Z}_p[[q^{1/N}]]).$$

Theorem 2 (B. 2026)

Let e_i be the coefficients of the polynomial $P(X) \cdot Q(X)$ as before. Then for every

$$f \in \mathrm{Im}(H^1_{\log-\mathrm{crys}}((\overline{X}, \overline{S})/\mathbb{Z}_p, \mathcal{H}_k) \to H^1_{\log-\mathrm{crys}}((\overline{X}, \overline{S})/\mathbb{Z}_p, \mathcal{H}_k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^1_{\mathrm{dR}}(V_\lambda, \mathcal{H}_k^{\mathrm{rig}}))$$

we have:

$$\sum_{i=0}^M e_{M-i} p^{M-i} a_{np^{l+i}}(f) \equiv 0 \mod p^{l(k+1)}.$$

An explicit example

Let $C/\mathbb{Q}$ be the CM elliptic curve $y^2 + xy = x^3 - x^2 - 2x - 1$, with CM by $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ and $j(C) = -3375$.

Zhang's basis for $\mathrm{Sym}^2 H^1_{\mathrm{dR}}(C) \hookrightarrow H^1_{\mathrm{dR}}(V_\lambda, \mathcal{H}_2^{\mathrm{rig}})$:

$$f_1 = \frac{E_4}{j+3375},$$
$$f_2 = 19 \cdot \frac{E_4}{j+3375} - 91125 \cdot \frac{E_4}{(j+3375)^2},$$
$$f_3 = 1399 \cdot \frac{E_4}{j+3375} - 19008675 \cdot \frac{E_4}{(j+3375)^2} + 54251268750 \cdot \frac{E_4}{(j+3375)^3}$$

For every ordinary prime $p > 3$ and $\ell \geq 0$, letting $u_p(C)$ be the p-adic unit root of $X^2 - a_p(C)X + p$:

$$a_{np^{\ell+1}}(f_i) \equiv \mu_i \cdot a_{np^\ell}(f_i) \pmod{p^{3\ell}}$$

where $\mu_1 = u_p(C)^2$, $\mu_2 = p$, $\mu_3 = p^2 u_p(C)^{-2}$.

Cohomological interpretation. The three forms f_1, f_2, f_3 represent the Frobenius eigenspace $\mathrm{Sym}^2 H^1_{\mathrm{dR}}(C)[1]$ inside $H^1_{\mathrm{dR}}(V_\lambda, \mathcal{H}_2^{\mathrm{rig}})$, and the congruences follow directly from Theorem 2 applied to $P(X) = \det(\mathrm{Frob} - X \mid \mathrm{Sym}^2 H^1_{\mathrm{dR}}(C)[1])$.

Further directions: towards semistable reduction and Gross-Zagier conjecture

In a joint work with Hazem Hassan, we are extending the framework to the **semistable case** for a Shimura curve admitting **p-adic uniformization** in the sense of Čerednik-Drinfeld. The residue sequence at CM divisors produces natural extensions of filtered (φ, N)-modules. We provide a p-adic analytic formula for the p-adic height pairing of these extensions that can be regarded as a **higher p-adic Green's function**. We expect the value to be the p-adic logarithm of an algebraic number, a p-adic analogue of the Gross-Zagier formula.

References

[BC94] F. Baldassarri and B. Chiarellotto.
Algebraic versus rigid cohomology with logarithmic coefficients.
In *Barsotti Symposium in Algebraic Geometry*, volume 15 of *Perspectives in Mathematics*, pages 11–50. Academic Press, 1994.

[CI10] Robert Coleman and Adrian Iovita.
Hidden structures on semistable curves.
In *Représentations p-adiques de groupes p-adiques III : méthodes globales et géométriques*, number 331 in Astérisque, pages 179–254. Société mathématique de France, 2010.

[Col96] R. F. Coleman.
Classical and overconvergent modular forms.
Inventiones Mathematicae, 124:215–241, 1996.

[Kat73] N. M. Katz.
P-adic properties of modular schemes and modular forms.
In Willem Kuijk and Jean-Pierre Serre, editors, *Modular Functions of One Variable III*, pages 69–190. Springer Berlin Heidelberg, Berlin, Heidelberg, 1973.

[Zha25] P. Zhang.
Elliptic curves and Fourier coefficients of meromorphic modular forms, 2025.