POINTWISE TO GLOBAL GOOD REDUCTION OF FAMILIES

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INTRODUCTION

Fix $p > 2$ a prime number, let $F/\mathbb{Q}_p$ be a p-adic field with valuation ring $\mathcal{O}_F$. Let $\mathcal{X}/\mathcal{O}_F$ be a smooth, separated scheme of finite type, let X/F denote its generic fiber with set of closed points $|X|$. Consider a smooth, proper family $Y \to X$ over F.

We assume that the family has *pointwise good reduction* (PGR), i.e.:

$\forall x \in |X|$[a], $Y_x/F(x)$ has a smooth, proper model $\mathcal{Y}_x/\mathcal{O}_{F(x)}$ making the following square cartesian:

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] & \mathcal{Y}_x \arrow[d] \\ F(x) \arrow[r] & \mathcal{O}_{F(x)} \end{tikzcd} \tag{PGR}$$

We study whether (PGR) implies that $Y \to X$ has *global good reduction* (GGR) over $\mathcal{X}$, i.e. that

$Y \to X$ has a smooth, proper model $\mathcal{Y}/\mathcal{X}$ making the following square cartesian:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ X \arrow[r] & \mathcal{X} \end{tikzcd} \tag{GGR}$$

The question of (PGR) $\implies$ (GGR) is asked in [1] as an application of their main result, and is answered positively for pointed smooth proper curves, and for abelian schemes under the ramification assumption $e(F/\mathbb{Q}_p) \leq p-1$ for the latter. With a different strategy, we partly recover and extend these results:

Theorem 1 ([3], Theorem 1.1)**.** *Assume that $Y \to X$ is either:*

- *(a) a smooth, proper family of genus g, n-pointed, geometrically connected curves with $2-2g < n$,*
- *(b) ($F/\mathbb{Q}_p$ unramified) a polarized abelian scheme,*
- *(c) ($F/\mathbb{Q}_p$ unramified) a primitively polarized K3 surface of degree 2d not divisible by p,*
- *(d) ($F/\mathbb{Q}_p$ unramified) a smooth, proper family of cubic fourfolds.*

Then (PGR) $\implies$ (GGR).

[a] We should replace $|X|$ with the subset $\mathrm{im}(\mathcal{X}(\overline{\mathcal{O}_F}) \to |X|)$ the set of all closed point admitting a specialization to the special fiber of $\mathcal{X}$. We omit this for simplicity.

MODULI REFORMULATION

Assume $Y \to X$ is as in Theorem 1. Then there exists a well-behaved (i.e. Deligne-Mumford, separated) moduli stack $\mathscr{M}/\mathcal{O}_F$, with generic fiber M/F, so that $Y \to X$ corresponds to a map $f : X \to M$ over F. There exists a global model $\mathcal{Y} \to \mathcal{X}$ if and only if f extends to a map $\mathcal{X} \to \mathscr{M}$ over $\mathcal{O}_F$.

Definition 2. We call f a *PGR morphism* if the family $Y \to X$ satisfies (PGR). We say that $\mathscr{M}$ has *PGR extension* if any PGR-morphism $f : X \to M$ extends to a morphism $\mathcal{X} \to \mathscr{M}$.

Now (PGR) $\implies$ (GGR) amounts to $\mathscr{M}$ having PGR-extension.

SMOOTH EXTENSION FOR SHIMURA STACKS

To show $\mathscr{M}$ has PGR-extension, we relate it to Shimura stacks for which certain types of extension properties hold. Let (G, Ω) be a Shimura datum with reflex field $\mathbb{Q}$, let $\mathrm{Sh}_K/\mathbb{Q}_p$ denote the Shimura stack of level K for any compact open subgroup $K \subseteq G(\mathbb{A}_f)$. We set

$$\mathrm{Sh}_\infty := \varprojlim_K \mathrm{Sh}_K.$$

Definition 3. A *smooth integral canonical model*[a] $\mathscr{S}_\infty/\mathbb{Z}_p$ of Sh_∞ is a smooth model with Hecke action, satisfying the *smooth extension property*:

- For all regular, formally smooth scheme $T/\mathbb{Z}_p$, any map
$$T \times_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{Sh}_\infty$$
uniquely extends to a map $T \to \mathscr{S}_\infty$.

[a] This definition is both incomplete and incorrect for the sake of simplicity.

Such models are known to exist for Shimura varieties of interest to us. ([5], Theorem 2.10], [4, Proposition 7.9]) Any K as above acts on $\mathscr{S}_\infty$, so we set

$$\mathscr{S}_K := \mathscr{S}_\infty/K.$$

TRANSFERING EXTENSION PROPERTIES

The precise relation between $\mathscr{M}$ and Shimura stacks is as follows: there exists a finite étale torsor $\mathcal{U} \to \mathscr{M}$ which admits an étale morphism $\mathcal{U} \to \mathscr{S}_K$ to a finite-level smooth integral canonical model of a Shimura stack.

Proposition 4. *In the diagram below, the extension properties of stacks imply each other from top-right to bottom-left:*

$$\begin{tikzcd} & & \mathscr{S}_\infty \arrow[d, "projection"] & \textit{smooth-extension} \\ \textit{PGR-extension} & \mathcal{U} \arrow[r, "étale"] \arrow[d, "finite\ torsor"'] & \mathscr{S}_K & \textit{PGR-extension} \\ \textit{PGR-extension} & \mathscr{M}. & & \end{tikzcd}$$

The key input for top-right to middle-right and for middle-left to bottom-left is [1, Theorem 1]; the remaining implication is established directly. Theorem 1, (b), (c) and (d) follow from Proposition 4. For (a), the key input is [6] showing directly that $\mathscr{M}$ has PGR-extension.

COUNTEREXAMPLE FOR ABELIAN SCHEMES

The ramification assumption $e(F/\mathbb{Q}_p) \leq p-1$ in [1] is optimal. To find a counterexample, we build upon a construction of Raynaud-Ogus-Gabber in [2, §6], later generalized by Vasiu-Zink in [7, Theorem 28].

Proposition 5 ([3], Proposition 3.7)**.** *Suppose $F/\mathbb{Q}_p$ has ramification index at least p. Then there exists a smooth local $\mathcal{O}_F$-algebra R with maximal ideal $\mathfrak{m}$ and an abelian scheme $A \to U := \operatorname{Spec} R \setminus \{\mathfrak{m}\}$ which does not extend to an abelian scheme over $\operatorname{Spec} R$.*

With notations from Proposition 5, we set $\mathcal{X} = \operatorname{Spec} R/\mathcal{O}_F$ and let X/F be its generic fiber. If $Y \to X$ denotes the restriction of the abelian scheme $A \to U$ to X, then one can show:

Corollary 6 ([3], Corollary 3.8)**.** *$Y \to X$ satisfies (PGR), but not (GGR).*

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