On the Semistability Defect of Hyperelliptic Curves and Cluster Pictures

University of Bristol
School of Mathematics

Authors: Allan Pérez & Jordi Vilà-Casadevall (joint with G. Jakovác, J. Martínez-Marín & J. Shi)

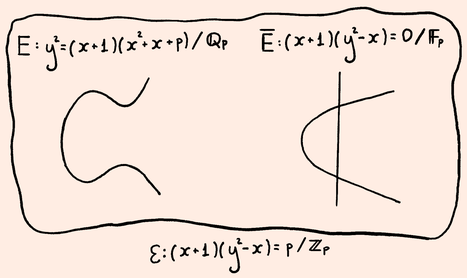
Semistable Reduction and the Monodromy Extension

Def: A *model* of a curve $C/\mathbb{Q}_p$ is a nice scheme $\mathcal{C}/\mathbb{Z}_p$ whose generic fiber is C, and whose special fiber is a (possibly reducible and non-reduced) curve $\overline{C}/\mathbb{F}_p$.

Def: A curve is said to have *semistable reduction* if there exists a model $\mathcal{C}$ with $\overline{C}$ a curve whose only singularities are ordinary double points.

Example (elliptic curves)

For elliptic curves, semistable reduction corresponds to good or multiplicative reduction. The following example illustrates a model of a curve with multiplicative reduction:



Remark: Semistable reduction of curves is a proprety Mumford developed in his GIT. By the celebrated theorem of Deligne and Mumford, this is equivalent to the semistability of the Galois representation associated to the curve $C/\mathbb{Q}_p$, namely an ℓ-adic Tate module of its Jacobian. By a result of Grothendieck on Galois representations, we have the following well-known fact.

Fact: Given $C/\mathbb{Q}_p^{\mathrm{nr}}$ there exists a unique minimal (finite and Galois) extension $K/\mathbb{Q}_p^{\mathrm{nr}}$ over which C_K becomes semistable. This extension is known as the **monodromy extension**. Its Galois group $\mathrm{Gal}(K/\mathbb{Q}_p^{\mathrm{nr}})$ is called the **monodromy group** of C, usually denoted by Φ.

Goal

Given a curve $C/\mathbb{Q}_p$, determine Φ as an abstract finite group.

References

[1] Tim Dokchitser et al. "Arithmetic of hyperelliptic curves over local fields". English. In: *Mathematische Annalen* 385 (Feb. 2022), pp. 1213–1322. ISSN: 0025-5831.

[2] Matthew Bisatt. "Clusters, inertia, and root numbers". English. In: *Functiones et Approximatio, Commentarii Mathematici* 66.2 (June 2022), pp. 203–243. ISSN: 0208-6573.

Cluster Pictures

Setting: We work with genus-g hyperelliptic curves

$$C : y^2 = f(x) = c_f \prod_{\alpha \in \mathcal{R}_f} (x - \alpha) \,/\mathbb{Q}_p^{\mathrm{nr}},$$

with $\deg(f) = 2g+1$ or $2g+2$ and $p \neq 2$.

Def: A *cluster* $\mathfrak{s}$ is a non-empty subset of $\mathcal{R}_f$ of the form $D \cap \mathcal{R}_f$ for some p-adic disc D. The *depth* of a cluster $\mathfrak{s}$ is

$$d_{\mathfrak{s}} := \min_{\alpha,\beta} v(\alpha - \beta).$$

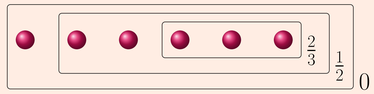
A *cluster picture* is a diagram where the roots of f are represented by dots and the clusters by circles around them, where the depth of each cluster is indicated.

Example (cluster picture)

Let $C : y^2 = (x-1)(x^2-p)(x^3-p^2)\,/\mathbb{Q}_p^{\mathrm{nr}}$, $p \neq 2, 3$. Then

$$\mathcal{R}_f := \left\{1,\ \pm p^{\frac{1}{2}},\ p^{\frac{2}{3}},\ \zeta_3 p^{\frac{2}{3}},\ \zeta_3^2 p^{\frac{2}{3}}\right\}$$

Computing the distance between the roots we get the following cluster picture:



Thm ([1]) $\mathrm{Jac}(C)$ has good reduction $\Leftrightarrow$

- $\forall\, \mathfrak{s} \neq \mathcal{R}_f$, $|\mathfrak{s}|$ is odd,
- $\mathcal{R}_f \subseteq \mathbb{Q}_p^{\mathrm{nr}}$, and
- $\forall$ principal cluster $\mathfrak{s}$, $\nu_{\mathfrak{s}} := v(c_f) + \sum_{\alpha} d_{\alpha \wedge \mathfrak{s}} \in 2\mathbb{Z}$.

A Criterion for the Tame Monodromy Group

Given $C/\mathbb{Q}_p^{\mathrm{nr}}$, write $|\Phi| = n$ (the monodromy degree). We say that C has *tame reduction* if p does not divide n, and *wild reduction* otherwise. By ramification theory of local fields, in the tame case,

$$\Phi \simeq C_n.$$

Theorem ([1], [2]): Assume $\mathrm{Jac}(C)/\mathbb{Q}_p^{\mathrm{nr}}$ as potentially good and tame reduction. We have that

$$n = 2^{\delta} \operatorname*{lcm}_{\mathfrak{s} \in \mathcal{R}_f} \mathrm{denom}\, d_{\mathfrak{s}}.$$

Here $\delta = 0$ if $n\nu_{\mathfrak{s}}$ is even for every principal cluster $\mathfrak{s}$, and $\delta = 1$ otherwise.

Remark: When C is an elliptic curve with minimal discriminant Δ_{min}, Kraus showed that n can be read off of the denominator of $v(\Delta_{min})$. The above result is a natural extension of this to hyperelliptic curves.

A Criterion for the Wild Monodromy Group

Theorem (Jakovác, Martínez-Marín, Shi, P, V-C)

Assume $p > g+1$ and $\mathrm{Jac}(C)/\mathbb{Q}_p^{\mathrm{nr}}$ has potentially good and wild reduction. Then, there is a unique cluster $\mathfrak{s}_p$ of size p which has

$$\mathrm{denom}(\mathfrak{s}_p) = s \text{ or } s \cdot p$$

for some $s \mid p-1$. We also have that

$$\operatorname*{lcm}_{\mathfrak{s}} \{\mathrm{denom}(d_{\mathfrak{s}})\} = r \text{ or } r \cdot p.$$

for some r coprime to p. Then

$$\Phi \simeq C_p \rtimes C_{2^{\delta} r},$$

where the quotient group acts via the faithful action of C_s on C_p. Here $\delta = 0$ if $rp\nu_{\mathfrak{s}}$ is even for every principal cluster $\mathfrak{s}$, and $\delta = 1$ otherwise.

The cluster picture itself is not enough to compute the monodromy group without knowing whether the curve has tame or wild reduction. For instance, both the curve

$$C_1 : y^2 = (x-10)(x-5)x(x+5)(x+10)/\mathbb{Q}_5^{\mathrm{nr}}, \text{ and}$$
$$C_2 : y^2 = x^5 - 625x^4 - 2500x^3 - 3750x^2 - 2500x - 625/\mathbb{Q}_5^{\mathrm{nr}},$$

have cluster picture

However, C_1 has tame reduction, thus $\Phi \cong C_2$. On the other side, C_2 has wild reduction, and the above theorem shows that $\Phi \cong C_{10}$

Remark Whether the curve has tame or wild reduction can be determined by the factorization of the polynomial $f(x)$, or from the special fiber of C by a theorem of T. Saito.

An implication for abelian varieties

These techniques can be extended to give a classification of the abstract finite groups Φ for hyperelliptic curves at certain primes (that depend on the genus). In particular, since genus 2 curves are all hyperelliptic, we get a classification of Φ for 2-dimensional simple PPAV (at odd primes). In 2004, Silverberg and Zarhin classified all Φ for abelian surfaces. Geometrically, these lists should coincide.

Theorem (Jakovác, Martínez-Marín, Shi, P, V-C)

There are abelian surfaces over a 3-adic field K with Φ not isomorphic to the monodromy group of any genus-2 Jacobian over K.

This phenomenon does not happen in any other p-adic field with $p > 3$.