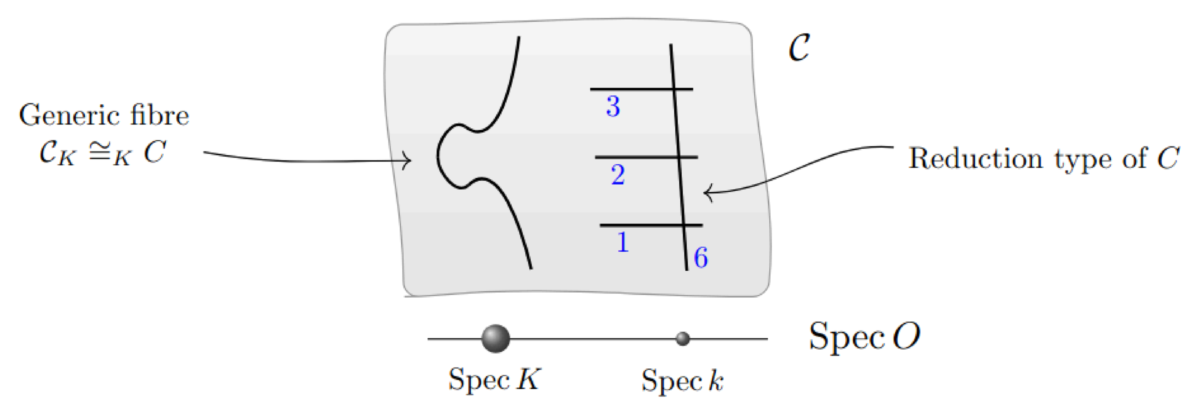
Reduction types of curves in tame extensions

Júlia Martínez-Marín

University of Bristol

Reduction types of curves

The **reduction type** of a curve C over a discretely valued field K encodes the discrete invariants of the special fibre of the minimal regular model with normal crossings (mrnc) of C.



Question A: How does the reduction type of a curve C/K change under base change by a tamely ramified extension L/K?

- If C/K is an elliptic curve, the reduction type of its base change C_L is determined by its reduction type over K and the degree $[L : K]$ (see [1]).
- If $g(C) = 2$, there are 2 reduction types that can become 2 different reduction types after a tame quadratic extension (see Examples a and b).

Question B: When is the reduction type of C_L/L uniquely determined by the reduction type of C/K and $[L : K]$?

Weight

The **weight of a component** Γ of multiplicity m in a reduction type is

$$\text{weight}(\Gamma) = \gcd(m, d_1, \ldots, d_r),$$

where $d_1, \ldots, d_r$ are the multiplicities of the components intersecting Γ.

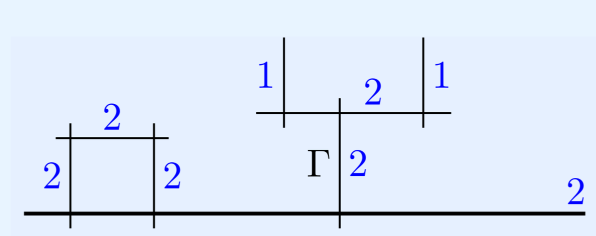
The **weight of a subset** $S = \{\Gamma_1, \ldots, \Gamma_r\}$ of components in a reduction type is

$$\text{weight}(S) = \gcd(\text{weight}(\Gamma_1), \ldots, \text{weight}(\Gamma_r)).$$

Examples

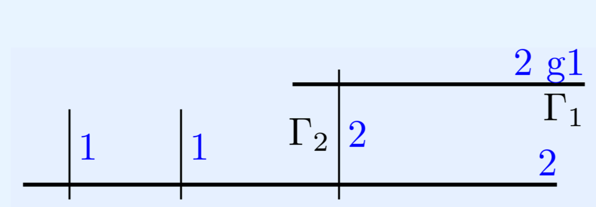
The examples below are reduction types of curves of genus 2, 2 and 3. Let Z denote the only cycle in each reduction type below, if there is one.

(a)



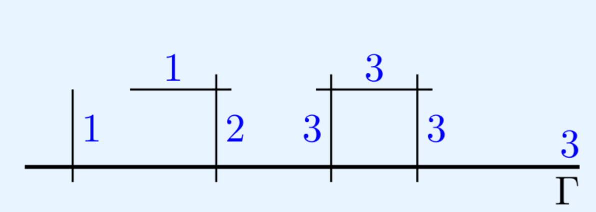
weight(Γ) = weight$(Z) = 2$, and the other components all have weight 1.

(b)



weight(Γ_1) = weight$(\Gamma_2) = 2$, and the other components all have weight 1.

(c)



weight(Z) = weight$(\Gamma) = 1$, even though the components in Z have mult=3.

Notation: (irreducible) components have their multiplicities in blue and their geometric genus g is omitted whenever it is 0.

Main results

We answer Question B for curves of arbitrary genus $g \geq 1$.

Let $p = \text{char}\, k$, where k is the residue field of K.

Theorem (MM. 2026+ [4])

Let $n \neq p$ be a prime. Then, a reduction type R has a unique *reduction type covering* of degree n if and only if one of the following holds.

1. Any component Γ with genus $g(\Gamma) \geq 1$ and any cycle Z in R satisfy
$$\gcd(\text{weight}(\Gamma), n) = 1 = \gcd(\text{weight}(Z), n).$$
2. R has exactly one component with genus $g \geq 1$ and no cycles, and
$$\gcd(\text{weight}(R), n) = n.$$
3. R has exactly one cycle and no vertices with genus $g \geq 1$ and
$$\gcd(\text{weight}(R), n) = n.$$

Corollary (MM. 2026+ [4])

Let $n \neq p$ be a prime. Suppose that a reduction type R satisfies one of 1, 2 or 3 above. If a curve has reduction type R, then its reduction type after base change by an extension of degree n is uniquely determined by R and n.

Some remarks

1. The corollary holds, in particular, for reduction types of curves with abelian and toric ranks $a = t = 0$.
2. Condition 1 holds for 102 out of 104 families of reduction types of genus 2 curves. The only types for which it doesn't hold are Examples a and b.
3. Conditions 2 and 3 occur more rarely, and not at all for curves of genus 2.

Principal components

A **principal component** in a reduction type is an irreducible component Γ satisfying either:

- its geometric genus is $g(\Gamma) \geq 1$, or
- Γ intersects at least 3 other components.

Other consequences

Using similar methods we get a new proof of a non-realisability result and a theorem characterising tame curves (i.e. curves that attain semistable reduction over a tame extension).

Corollary (Liu-Lorenzini-Raynaud [3])

Let C/K be a geometrically connected curve with abelian and toric ranks

$$a_C = t_C = 0.$$

Let $d_1, \ldots, d_r$ denote the multiplicities of every component in the reduction type of C. If $d := \gcd(d_1, \ldots, d_r) > 1$, then $\text{char}\, k = p > 0$ and $d = p^s$ for some $s \geq 1$.

Theorem (MM. 2026+ [4])

A tame curve has potentially good reduction if and only if its reduction type has only one principal component and no loops.

Computing reduction type coverings

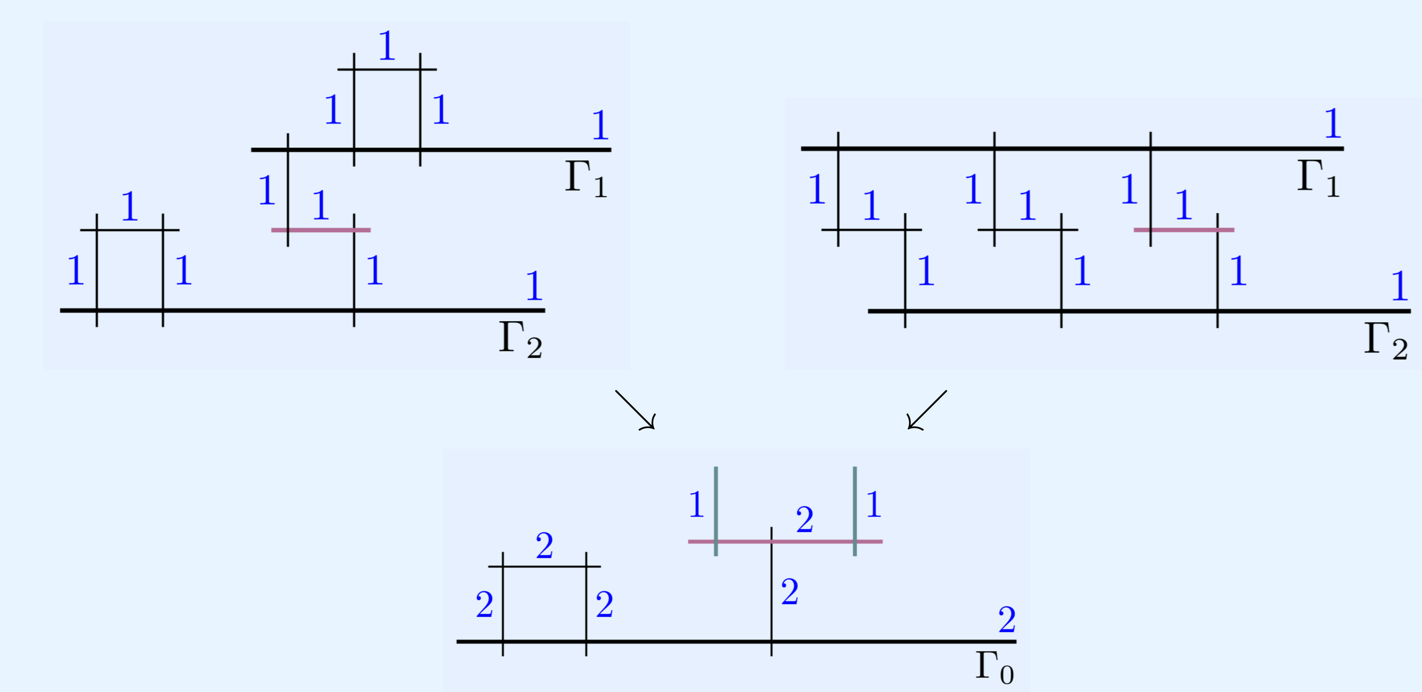
We answer Question A for curves of arbitrary genus $g \geq 1$ and $n = [L : K]$ prime.

- **Locally:** multiplicities and number of components lying above each component follow from explicit computations in [2], and together with Riemann-Hurwitz one can compute the genera.
- **Globally:** look at branched coverings of the dual graph with ramification determined by the weight of each component.

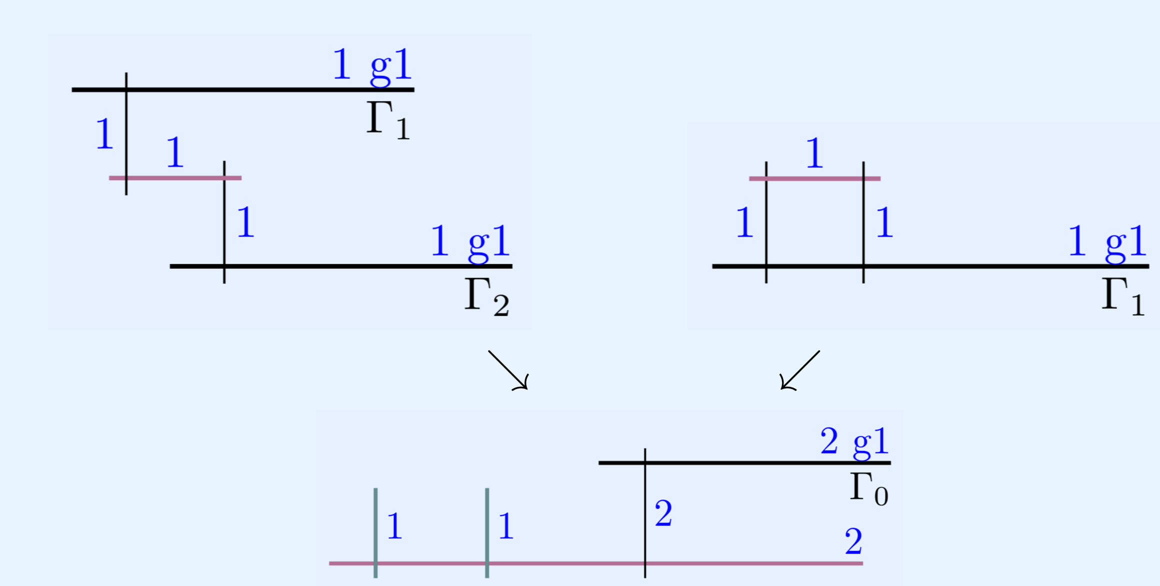
Examples

The following examples are the 2 reduction types in genus 2 for which none of 1, 2 or 3 of the theorem hold. We construct their reduction type coverings of degree 2.

(a)



(b)



Notation: Γ_1 and Γ_2 map to Γ_0. The non-black components in the initial reduction type have weight coprime to 2, so they are in the branching locus. The components in green get contracted after a quadratic extension.

Work in progress

Extend the main results to composite n (this is non-trivial!).

References

[1] Tim Dokchitser and Vladimir Dokchitser. A remark on tate's algorithm and kodaira types. *Acta Arithmetica*, 160(1):95–100, 2013.

[2] Lars Halvard Halle and Johannes Nicaise. *Néron Models and Base Change.* Springer Cham, 2016.

[3] Qing Liu, Dino Lorenzini, and Michel Raynaud. Néron models, lie algebras, and reduction of curves of genus one. *Inventiones mathematicae*, 157(3), 2004.

[4] Júlia Martínez-Marín. Reduction types of curves and tame ramification. *Manuscript in preparation.*

julia.martinezmarin@bristol.ac.uk