Counting Rational points on Smooth Surfaces and Threefolds

Lorenzo Andreaus (Université Paris Cité)

Université Paris Cité

INSTITUT DE MATHÉMATIQUES
JUSSIEU - PARIS RIVE GAUCHE

Context

- Assuming a suitable hypothesis on the size of the rank of elliptic curves over the rationals, Heath-Brown [Hea98] shows an asymptotic upper bound for the number of rational points of bounded height on a cubic surface which do not lie on any rational line contained in the surface.
- Dimitrov, Gao and Habegger [DGH21] recently proved a version of Faltings's Theorem which is uniform on the genus and on the rank of the Jacobian of the curve.
- Using Heath-Brown's approach and the Uniform Faltings's Theorem as a technical tool, we show that, assuming a suitable hypothesis on the size of the rank of Abelian Varieties over number fields, we obtain an asymptotic upper bound for the number of rational points of bounded height on a surface of degree $d \geq 4$ which do not lie on lines. For $d = 4$ and $d = 5$, this improves an unconditional result from Salberger [Sal23].

Heights on Projective Spaces

- Let K be a number field. For any place v of K, let $\|\cdot\|_v$ denote the normalized absolute value associated with v.
- We define the **height function** in $\mathbb{P}^n$ by $H_K([x_0\colon \cdots \colon x_n]) = \prod_v \max\{\|x_0\|_v, \dots \|x_n\|_v\}$.
- **Northcott Property**: for any $B > 0$, there is a finite number of $x \in \mathbb{P}^n(K)$ with $H_K(x) \leq B$.

Counting Rational Points on Projective Varieties

- Given a projective variety $X \subseteq \mathbb{P}^n$ over a number field K, we define $N_X(B) \coloneqq \#\{x \in X : H_K(x) \leq B\}$. We wish to find asymptotics for $N_X(B)$.
- Let $X' \subseteq X$ be the complement of the union of all lines contained in X. In some cases, we want to study $N_{X'}(B)$.
- Schanuel [Sch79] shows that $N_{\mathbb{P}^n}(B) \sim aB^{n+1}$, where $a > 0$ is an explicit constant depending only on K and n.
- Browning, Heath-Brown and Salberger [BHS06], using the *determinant method*, show that, for geometrically integral varieties of degree $d = 2$ and $d \geq 6$, $N_X(B) \ll_{n,d,\varepsilon} B^{\dim X + \varepsilon}$.
- Salberger [Sal23] shows that if $X \subseteq \mathbb{P}^n$ is an integral projective variety of degree $d \geq 2$ over $\mathbb{Q}$, then $N_X(B) \ll_{X,\varepsilon} B^{\dim X + \varepsilon}$. He also shows that $N_X(B) \ll_{n,d,\varepsilon} B^{\dim X + \varepsilon}$ for all $d \geq 4$, and $N_X(B) \ll_{n,\varepsilon} B^{\dim X - 1 + 2/\sqrt{3} + \varepsilon}$ if $d = 3$. Paredes and Sasyk [PS22] later showed that the same holds for any number field K, if we allow the implicit constant to depend on K. It is conjectured that the bound also holds for $d = 3$.

Counting Rational Points on Smooth Hypersurfaces

- If $X \subseteq \mathbb{P}^n$ is a smooth hypersurface of degree $n \geq 4$ and $d \geq 3$, it is conjectured that $N_X(B) \ll_{n,d,\varepsilon} B^{n-2+\varepsilon}$. Marmon [Mar10] shows this for $n \geq 28$, using the *circle method*. and Verzobio [Ver25] shows this for $n \geq 4$ and $d \geq 50$ using the determinant method. Verzobio also finds other bounds for $d \geq 6$.
- For $n = 3$, this bound does not hold if X contains a line defined over K, since in this case $N_X(B) \gg B^2$. Therefore, when $X \subseteq \mathbb{P}^3$ is a smooth surface, we are interested in counting $N_{X'}(B)$.
- Salberger [Sal23] shows using the determinant method that, for $K = \mathbb{Q}$, $N_{X'}(B) \ll_d B^{3/\sqrt{d}}(\log B)^4 + B$.
- Assuming the **Rank Hypothesis** for elliptic curves over $\mathbb{Q}$, Heath-Brown shows that, if X is a smooth cubic surface, $N_{X'}(B) \ll_{X,\varepsilon} B^{4/3+\varepsilon}$. His idea is to cut X by projective planes and bound the number of points on each of the curves of the intersections.

Hypothesis 1: Rank Hypothesis

For any K-Abelian variety A, let N_A denote its conductor (that is, the ideal norm of the conductor ideal of A) and let r_A denote its rank. Then we have $r_A = o(\log N_A)$ as $N_A \to \infty$.

The Rank of Abelian Varieties

- Assuming modularity for the Abelian variety A over the number field K, this hypothesis would hold for A by a combination of the Generalized Riemann Hypothesis and the Birch and Swinnerton-Dyer conjecture (see [And26]).
- Ooe and Top [OT89] show that, for an Abelian variety A/K we have, unconditionally, $r_A = O_{K,\dim A}(N_A)$.
- In the case of elliptic curves, more evidence is known. All elliptic curves over the rational number are modular by [BCDT01]. If we assume that an elliptic curve $E/\mathbb{Q}$ has at least one rational point of order 2, then Heath-Brown [Hea98] shows that $r_E = O\left(\frac{\log N_E}{\log\log N_E}\right) = o(\log N_E)$.
- What we actually need is the a priori weaker bound $r_A = o(h_{\mathrm{Falt}}(A))$, where $h_{\mathrm{Falt}}(A)$ denotes the Faltings's height of A.

Theorem 2: A Bound for Rational Points on Surfaces in $\mathbb{P}^3$ [And26]

Assume that the Rank Hypothesis holds. Let $X \subseteq \mathbb{P}^3$ be a smooth surface of degree $d \geq 4$ over a number field K, and let $\varepsilon > 0$. Then we have $N_{X'}(B) \ll_{K,d,\varepsilon} B^{4/3+\varepsilon}$.

The result and the idea behind its proof

- The theorem above improves Salberger's bound for $d = 4$ and $d = 5$.
- To prove this theorem, we cover the points of bounded height with projective planes of bounded height in the dual projective space $(\mathbb{P}^3)^\vee$. This can be done using a result from Schmidt [Sch67].
- We then apply Schanuel's result [Sch79] to bound the number of such planes asymptotically, and proceed by estimating the number of points of bounded height in the intersection between X and each of these planes.
- This intersection will be generically a smooth curve. In this case, we use a uniform version of Faltings's Theorem, due to Dimitrov, Gao, and Habegger.
- We bound the number of rational points on curves of (geometric) genus at least 2 that appear as irreducible components of the intersections between X and rational planes by applying the Uniform Falting's Theorem to their normalizations.
- We can proceed similarly to bound the number of rational points on the irreducible components of genus 1, by using a bound on the number of smooth cubic plane curves due to Heath-Brown [Hea98], which is generalized to global fields by Glas and Hochfilzer [GH24].
- Finally, we show that the curves of genus 0 that appear do not alter our estimation. We remove dependence on X by applying a standard trick due to Heath-Brown.

Theorem 3: Uniform Faltings's Theorem [DGH21]

For each integer $g \geq 2$, there exists a constant $c(g) > 0$ depending on $[K\colon \mathbb{Q}]$ such that, for every smooth projective curve C over K of (geometric) genus g, we have $\#C(K) \leq c(g)^{1+\rho(C)}$, where $\rho(C)$ denotes the rank of $\mathrm{Jac}(C)(K)$.

Covering by planes

- A projective plane $\Pi\colon a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0$ contained in $\mathbb{P}^3$ can be identified with the point $[a_0\colon a_1\colon a_2\colon a_3] \in (\mathbb{P}^3)^\vee$, and we define $H_K(\Pi)$ as the height of this point.
- A result from Schmidt [Sch67] shows that, for every point $x \in \mathbb{P}^3$, we can find a plane $\Pi \subseteq \mathbb{P}^3$ such that $H_K(\Pi) \ll_K H_K(x)^{1/3}$.
- By Schanuel [Sch79], $N_{(\mathbb{P}^3)^\vee}(B^{1/3}) \ll_K B^{4/3}$, so we can cover all the points on X of height smaller than or equal to B by $O_K(B^{4/3})$ projective planes.

Counting points on plane section components of genus $g \geq 1$

- Each $X \cap \Pi$ is a plane curve of degree d. Let $C \subseteq X \cap \Pi$ be an irreducible component of degree $e \leq d$.
- C has a number $O_d(1)$ of singularities, so it suffices to bound $N_{\tilde{C}}(B)$, where $\tilde{C}$ is the normalization of C.
- Using the Uniform Faltings's Theorem, a calculation shows that if $g(\tilde{C}) \geq 2$, $\#\tilde{C}(K) \ll_{K,d,\varepsilon} B^\varepsilon$, so these curves contribute with a total of $O_{K,d,\varepsilon}(B^{4/3+\varepsilon})$ points.
- If $\tilde{C}$ has genus 1, we can use a proposition from Glas-Hochfilzer [GH24] to conclude that $N_{\tilde{C}}(B) \ll_{K,d,\varepsilon} B^\varepsilon$, so that we also obtain a total contribution of $O_{K,d,\varepsilon}(B^{4/3+\varepsilon})$ points.

Proposition 4: Rational Points on Elliptic Curves [GH24]

Assume that the Rank Hypothesis holds. Let $E \subseteq \mathbb{P}^m$ be a smooth curve of genus 1. Then, for any $\varepsilon > 0$, we have $N_E(B) \ll_{K,d,m,\varepsilon} B^\varepsilon$.

Points lying on rational curves

- All that is left is to bound the number of rational points in curves of genus 0 that lie inside X. If $C \subseteq X$ is a curve of genus 0, then either $C(K) = \emptyset$ or C is a rational curve. It suffices to consider the second case.
- Since we are not counting points on lines, suppose $\deg C = e \geq 2$. Paredes ans Sasyk [PS22] show that $N_C(B) \ll_{K,d,\varepsilon} B^{2/e+\varepsilon} \leq B^{1+\varepsilon}$.
- Since the contribution of a single rational curve of degree $e \geq 2$ is less than $O_{K,d,\varepsilon}(B^{4/3+\varepsilon})$, and since we have d possibilities for e, the following lemma concludes our proof:

Lemma 5: Number of Rational Plane Curves of Fixed Degree [And26]

Let $X \subseteq \mathbb{P}^3$ be a smooth surface of degree $d \geq 4$. Fix a positive integer e. Then X contains finitely many rational plane curves of degree e. Furthermore, the number of such curves is $O_{d,e}(1)$.

Projective Surfaces in $\mathbb{P}^n$

- Suppose that $X \subseteq \mathbb{P}^n$ is a smooth surface.
- By Schmidt [Sch67], every point $x \in \mathbb{P}^n$ of height $H_K(x) \leq B$ is contained in a hyperplane $\Pi \subseteq \mathbb{P}^n$ such that $H_K(\Pi) \ll_K B^{1/n}$.
- By Schanuel [Sch79], this shows that we can cover all the point on X of height smaller than or equal to B by $O_K(B^{\frac{n+1}{n}})$ projective hyperplanes.
- If X is non-degenerate, each $X \cap \Pi$ is a curve of degree d. We can then proceed as in the proof of Theorem 2.
- If furthermore X is non-uniruled, we have the following lemma:

Lemma 6: Number of Rational Curves of Fixed Degree [And26]

Let $X \subseteq \mathbb{P}^n$ be a non-uniruled surface of degree d. Fix a positive integer e. Then X contains finitely many rational curves of degree e. Furthermore, the number of such curves is $O_{n,d,e}(1)$.

Theorem 7: A Bound for Rational Points on Surfaces in $\mathbb{P}^n$ [And26]

Assume that the Rank Hypothesis holds. Let $n \geq 3$, let $X \subseteq \mathbb{P}^n$ be a non-degenerate non-uniruled smooth surface of degree d, and let $\varepsilon > 0$. Then we have $N_{X'}(B) \ll_{K,n,d,\varepsilon} B^{\frac{n+1}{n}+\varepsilon}$.

Counting Rational Points on Smooth Threefolds (Work in Progress)

- Let $X \subseteq \mathbb{P}^4$ be a smooth threefold of degree $d \geq 4$ over a number field K.
- We can cut X by hyperplanes $\Lambda \subseteq \mathbb{P}^4$ and use Theorem 2 to bound the rational points of bounded height of X, using results from Salberger [Sal23] and [Sal05].

Theorem 8: A bound for rational points on Threefolds in $\mathbb{P}^4$

Assume the Rank Hypothesis holds. Let $X \subseteq \mathbb{P}^4$ be a smooth threefold of degree $d \geq 4$ over $\mathbb{Q}$, and let $\varepsilon > 0$. Then $N_X(B) \ll_{K,d,\varepsilon} B^{5/2+\varepsilon}$ if $d \geq 5$, and $N_{X'}(B) \ll_{K,\varepsilon} B^{5/2+\varepsilon}$ if $d = 4$.

Future Directions

- The theorem above holds over $\mathbb{Q}$, since the auxiliary results from Salberger we use are not written for a general number field. We plan to generalize these results to make the theorem above over K.
- Another future goal is to see if we can replicate the same method also for varieties of higher dimension.

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 lorenzo.andreaus@imj-prg.fr