Existence of minimal del Pezzo surfaces of degree 1 with conic bundles over finite fields

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Introduction

Del Pezzo surface X over a field k: smooth projective geometrically integral surface such that $-K_X$ is ample.

Degree: $d := K_X^2$, satisfies $1 \leq d \leq 9$. We consider $d = 1$.

If $k = \overline{k}$: del Pezzo surface of degree 1 is a blow-up of $\mathbb{P}^2$ in 8 points in general position.

If $k = \mathbb{F}_q$, we classify del Pezzo surfaces X based on the conjugacy class of the Frobenius automorphism in $\mathrm{Pic}(\overline{X})$. We call this the **type** of the surface.

For degree 1: 112 possible conjugacy classes [7], of which 37 are **minimal**, and 7 of those have a **conic bundle structure**.

Question

For which q does there exist a **minimal del Pezzo surface of degree 1 with conic bundle** of each of the 7 types over $\mathbb{F}_q$?

Classification

Standard conic bundle $f\colon X \to \mathbb{P}^1$ of degree 1 over $\overline{k}$: 7 singular fibers, which are pairs of (-1)-curves intersecting transversally in a point.

Over $\mathbb{F}_q$, the singular fibers lie above closed points, with the sum of their degrees equal to 7.

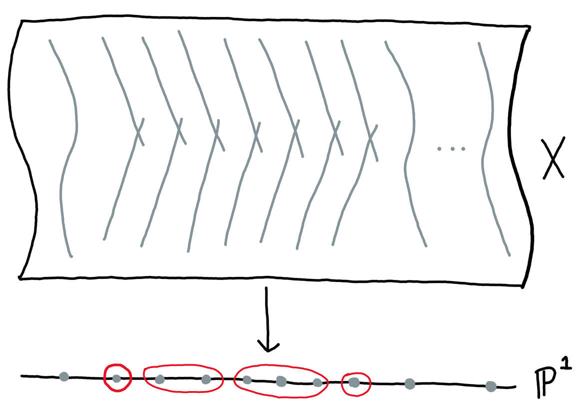


Figure 1. A standard rational conic bundle with 7 singular fibers.

Proposition 1 ([3]). *There exists a minimal standard conic bundle $f\colon X \to \mathbb{P}^1$ over $\mathbb{F}_q$, with singular fibers above closed points $x_1, \ldots, x_s \in \mathbb{P}^1_{\mathbb{F}_q}$, if and only if s is even.*

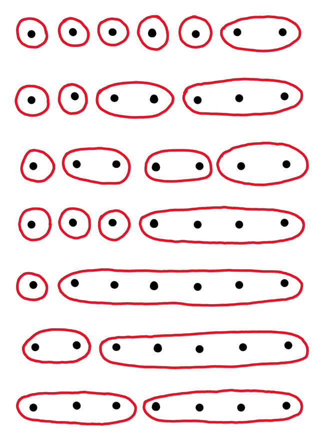
Degree 1: $x_1, \ldots, x_s \in \mathbb{P}^1_{\mathbb{F}_q}$ with s even, sum of degrees equal to 7.
There is a 1-1 correspondence:

Ways of partitioning 7 points into an even number of parts

$\updownarrow$

7 types of minimal del Pezzo surfaces of degree 1 with conic bundle

- Type 1: five $\mathbb{F}_q$-points, one point of degree 2;
- Type 2: two $\mathbb{F}_q$-points, one point of degree 2, one point of degree 3;
- Type 3: one $\mathbb{F}_q$-point, three points of degree 2;
- Type 4: three $\mathbb{F}_q$-points, one point of degree 4;
- Type 5: one $\mathbb{F}_q$-point, one point of degree 6;
- Type 6: one point of degree 2, one point of degree 5;
- Type 7: one point of degree 3, one point of degree 4.



Main result

There exists a del Pezzo surface of degree 1 over $\mathbb{F}_q$:

- *of type 1 if and only if $q \geq 5$.*
- *of type 2 if and only if $q \geq 3$.*
- *of type 3 if $q \geq 4$, and it does not exist when $q = 2$.*
- *of type 4 if $q \geq 7$.*
- *of type 5 if $q \geq 4$.*
- *of type 6 for all q.*
- *of type 7 for all q.*

Method 1

By Proposition 1, for the 7 types: If configuration of points exists in $\mathbb{P}^1_{\mathbb{F}_q}$, then conic bundle with singular fibers above those points exists.

Use existence of conic bundle to construct del Pezzo surface of same type, using method inspired by [5]. We use the following **elementary transformations** of $f\colon X \to \mathbb{P}^1$:

- Take $P \in X(\mathbb{F}_q)$ on smooth fiber F of f;
- Blow up X in P;
- Blow down strict transform of F.

The resulting surface is again a minimal standard conic bundle of the same type.

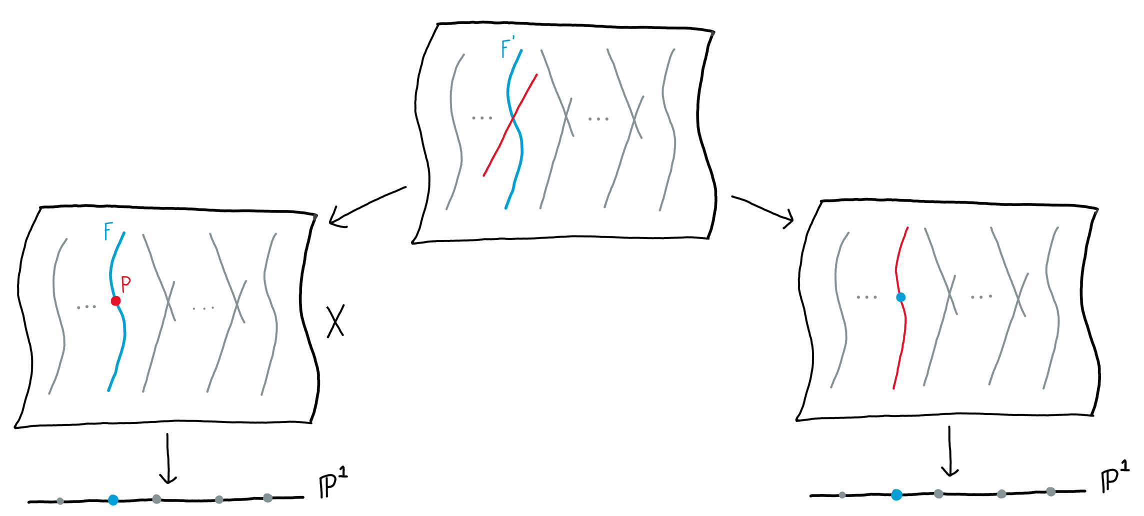
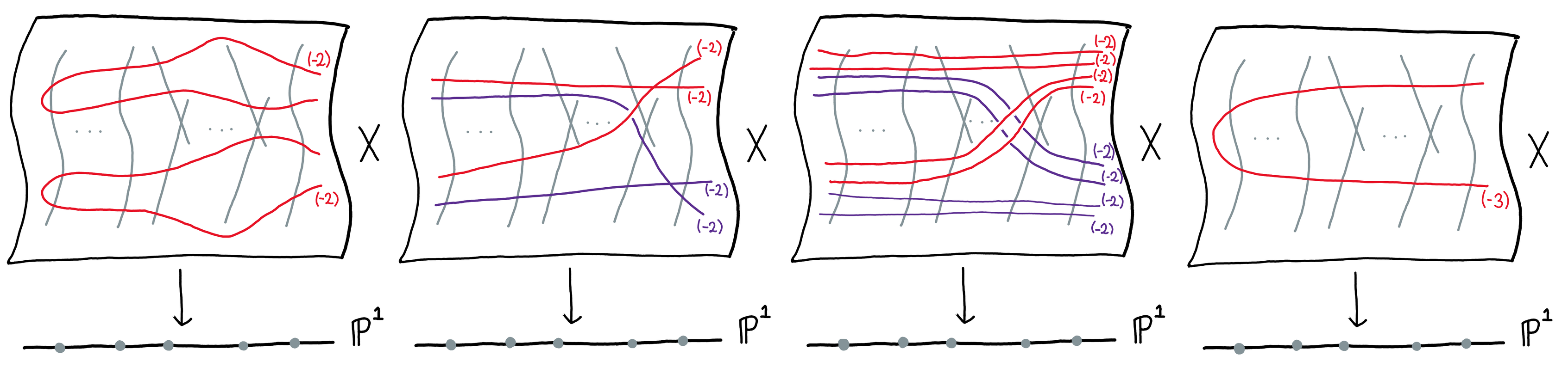


Figure 2. An elementary transformation: we blow up in P, and subsequently blow down F'.

Strategy

Proposition 2. *Let $f\colon X \to \mathbb{P}^1$ be a minimal standard conic bundle of degree 1 over a perfect field k. Then one of the following holds.*

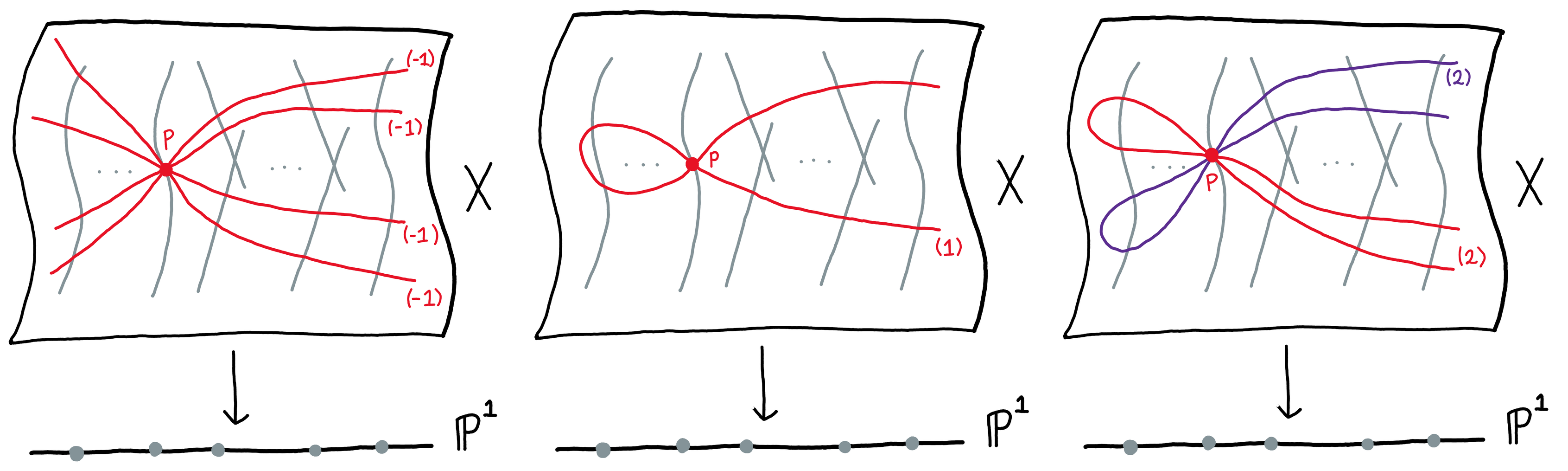
1. *The surface X is a **del Pezzo surface** of degree 1 with two conic bundle structures.*
2. *There are **two disjoint Galois-conjugate bisections** with self-intersection -2.*
3. *There are **four Galois-conjugate sections**, each with self-intersection -2.*
4. *There are **eight disjoint Galois-conjugate sections** with self-intersection -2.*
5. *There is an irreducible **bisection** on X with self-intersection -3.*



Case 2 Case 3 Case 4 Case 5

We use elementary transformations as in Figure 2 to transform a conic bundle over $\mathbb{F}_q$ of case 2, 3, 4, 5 into a del Pezzo surface (case 1), if q is large enough.

Strategy: For each case, show that there exists an $\mathbb{F}_q$-point P on a smooth fiber of f which avoids the following situations:



(a) An $\mathbb{F}_q$-point on sections with selfintersection -1.

(b) A singular $\mathbb{F}_q$-point on a bisection defined over $\mathbb{F}_q$ with self-intersection 1.

(c) an $\mathbb{F}_q$-point which is a singular point on two Galois-conjugate bisections with self-intersection 2.

Perform elementary transformation at $P \rightsquigarrow$ no more curves of self-intersection < -1.

For each case in Proposition 2, and each type, we give an explicit bound for the number of points to avoid per fiber, so if q is large enough, we can find an $\mathbb{F}_q$-point P on a smooth fiber which avoids situations (a), (b) and (c).

Method 2

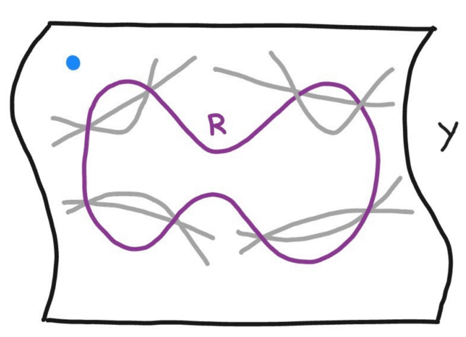
Each del Pezzo surface of degree 1 (dP1) over $\mathbb{F}_q$ has a unique quadratic twist called its **Bertini twist**, which is again a dP1 over $\mathbb{F}_q$.

Type 1, 2 and 4: Bertini twist has a (-1)-curve defined over $\mathbb{F}_q \rightsquigarrow$ can be blown down to del Pezzo surface of degree 2 (dP2).

We can derive the existence of such a dP1 from the existence of the corresponding dP2.

Existence of del Pezzo surfaces of degree 2 for each q is classified in [6].

- Let Y be a dP2 over $\mathbb{F}_q$.
- The morphism induced by $|-K_Y|$ has a ramification divisor R, which lies above a quartic curve in $\mathbb{P}^2$.
- Let $S := R \cup \{(-1)\text{-curves on } Y\}$.
- By [4]: $\mathrm{Bl}_P Y$ is dP1 $\iff P \notin S$.
- $\#Y(\mathbb{F}_q) = q^2 + a(Y)q + 1$, and



$$\#S(\mathbb{F}_q) \leq \begin{cases} 2(q+1) + \#\{\mathbb{F}_q\text{-points on } (-1)\text{-curves on } Y\} & \text{if } q \text{ even;} \\ (q+1+6\sqrt{q}) + \#\{\mathbb{F}_q\text{-points on } (-1)\text{-curves on } Y\} & \text{if } q \text{ odd.} \end{cases}$$

- If $\#Y(\mathbb{F}_q) - \#S(\mathbb{F}_q) \geq 1 \rightsquigarrow \exists\, \mathbb{F}_q$-point outside $S \rightsquigarrow$ dP1 exists.

For dP1 of index 8 (blow-up of $\mathbb{P}^2$): determine (non-)existence for small q computationally using Magma.

References

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