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The Beilinson–Bloch conjecture for some non-isotrivial varieties over global function fields

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Abstract

The Beilinson–Bloch conjecture is a generalization of the Birch and Swinnerton-Dyer conjecture, which relates the ranks of Chow groups of smooth projective varieties over global fields to the order of vanishing of L-functions. We prove the conjecture for certain classes of non-isotrivial varieties over $\mathbb{F}_q(t)$, including some cubic threefolds and fivefolds. We deduce the Birch and Swinnerton-Dyer conjecture for their intermediate Jacobians, and use it to establish new cases of the Tate conjecture over finite fields.

Conjectures on algebraic cycles

Let k be a field, and X a smooth projective variety over k. Let $\mathrm{CH}^i(X)$ denote the Chow group of codimension-i algebraic cycles in X. The Chow group is an important arithmetic invariant of X, whose structure is not well-understood in general.

One of the most fundamental open problems concerning Chow groups is the Tate conjecture [Tat65]. To state it, we fix a prime ℓ invertible in k.

Conjecture (Tate). *When k is a finitely generated field and $i \in \mathbb{Z}$, the ℓ-adic cycle class map*

$$\mathrm{cl} : \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^{2i}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell(i))^{G_k}$$

is surjective.

The Tate conjecture describes the image of the cycle class map, but says nothing about the kernel. We thus let

$$\mathrm{CH}^i(X)_0 = \ker(\mathrm{cl} : \mathrm{CH}^i(X) \to H^{2i}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell(i)))$$

denote the homologically trivial subgroup of $\mathrm{CH}^i(X)$. Over a global field, the rank of $\mathrm{CH}^i(X)_0$ is predicted by the Beilinson–Bloch conjecture ([Bei87], [Blo84]).

Conjecture (Beilinson, Bloch). *When k is a global field and $i \in \mathbb{Z}$, we have*

$$\mathrm{rk}\, \mathrm{CH}^i(X)_0 = \operatorname*{ord}_{s=i} L(H^{2i-1}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell), s).$$

For X an abelian variety and $i = 1$, the Beilinson–Bloch conjecture is equivalent to the rank part of the Birch and Swinnerton-Dyer conjecture for X. Unconditional evidence for the Beilinson–Bloch conjecture is sparse, particularly over number fields.

Beilinson–Bloch in positive characteristic

Tate [Tat65] and Artin–Tate [Tat66] recognized in the 1960s that the BSD conjecture over global function fields was closely related to the Tate conjecture over finite fields. Several later authors elaborated on the relationship between the two conjectures, including Jannsen [Jan90] and Geisser [Gei21]. In [Bro25] we refined some of Jannsen's results, and gave a general sufficient condition for the Beilinson–Bloch conjecture over a function field.

Prior to the present work, the Beilinson–Bloch conjecture had been proved for certain classes of isotrivial varieties over function fields ([Jan07], [Kah21]). Here a variety over a function field $K / \mathbb{F}_q$ is called isotrivial if its base change to $\overline{K}$ is defined over $\overline{\mathbb{F}}_q$. These earlier results suggested the following problem.

Problem. *Find interesting non-isotrivial varieties over function fields which satisfy the Beilinson–Bloch conjecture in codimension > 1.*

Non-isotrivial cases of Beilinson–Bloch

The criterion of [Bro25] for the Beilinson–Bloch conjecture requires some control over the bad reduction of a variety over a function field. It is thus natural to apply the criterion in examples where the singularities of the special fibers are as mild as possible.

Recall that a **Lefschetz pencil** for a smooth projective variety $X \subset \mathbb{P}^r_k$ is roughly a generically smooth family of hyperplane sections of X, parametrized by $\mathbb{P}^1_k$, such that each singular member of the family (over the algebraic closure) is smooth away from one ordinary double point singularity.

Theorem 1. *Let $X \subset \mathbb{P}^5_{\mathbb{F}_q}$ be a smooth cubic fourfold, and let $f : Y \to \mathbb{P}^1_{\mathbb{F}_q}$ be a Lefschetz pencil for X. Then the Beilinson–Bloch conjecture holds for the generic fiber Y_η of f.*

The variety Y_η is a smooth cubic threefold over $\mathbb{F}_q(t)$. The proof of Theorem 1 relies on the known Tate conjecture for smooth cubic fourfolds over finite fields, established in [Cha13], [Mad15], and [AKPW2025].

We additionally establish the Beilinson–Bloch conjecture for certain cubic fivefolds over $\mathbb{F}_q(t)$, and prove the finiteness of the Albanese kernel for a class of surfaces in $\mathbb{P}^3_{\mathbb{F}_q(t)}$. These surfaces can have arbitrary degree, subject to some congruence conditions. The varieties covered by these results are typically non-isotrivial: for instance, the cubic threefolds Y_η of Theorem 1 are never isotrivial.

BSD for the intermediate Jacobian

Theorem 1 turns out to imply a case of the Birch and Swinnerton-Dyer conjecture. Lines on a smooth cubic threefold are parametrized by a smooth projective surface, the **Fano surface**. The **intermediate Jacobian** of the cubic threefold is defined to be the Albanese variety of the Fano surface. This is a five-dimensional abelian variety canonically associated with the cubic threefold.

Let J be the intermediate Jacobian of the cubic threefold Y_η of Theorem 1.

Theorem 2. *The strong Birch and Swinnerton-Dyer conjecture holds for J. In particular, the leading nonzero Taylor coefficient of the L-function $L(J, s)$ at $s = 1$ is given by an explicit formula involving the order of the Tate–Shafarevich group of J.*

The proof uses Theorem 1 and a result of Kato–Trihan [KT03].

We also explicitly compute the L-function of J, in terms of the zeta function of the total space Y of the Lefschetz pencil. This yields examples (related to Fermat varieties) where J has Mordell–Weil rank 28. Moreover, the normalized leading term of the L-function in these examples is equal to one. This may indicate trivial values of all BSD invariants of J.

A case of the Tate conjecture

Let $Y_\eta / \mathbb{F}_q(t)$ be a cubic threefold as in Theorem 1. Let F_η be the Fano surface of Y_η, let $\mathcal{F} \to \mathbb{P}^1_{\mathbb{F}_q}$ be a "spreading out" of F_η, and let $\mathcal{F}'$ be a resolution of singularities of $\mathcal{F}$ [CP08]. This $\mathcal{F}'$ is a smooth projective threefold over $\mathbb{F}_q$.

Theorem 3. *The Tate conjecture holds for $\mathcal{F}'$.*

The proof combines three ingredients: Theorem 2, the known Tate conjecture for Fano surfaces of cubic threefolds ([Rou13], [DLR17]), and a theorem of Geisser [Gei21].

Sketch of proof of Theorem 1

Let $X \subset \mathbb{P}^5_{\mathbb{F}_q}$ be a smooth cubic fourfold, and let $f : Y \to \mathbb{P}^1_{\mathbb{F}_q}$ be a Lefschetz pencil of hyperplane sections of X. The idea of the proof of Theorem 1 is to control the Chow groups and ℓ-adic homology of Y, and the Chow groups and homology of the union Z of all singular fibers of f. We then use suitable localization exact sequences to control the Chow groups and homology of $Y - Z$. This ultimately yields control over corresponding invariants of the generic fiber of f.

A form of the classical Bloch–Srinivas argument ([BS83], [FV23]) shows that rational and numerical equivalence agree on $\mathrm{CH}^2(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. This implies that the ℓ-adic cycle class map is injective on $\mathrm{CH}^2(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$. It thus follows from the Tate conjecture for X that $\mathrm{CH}^2(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \cong H^4(\overline{X}, \mathbb{Q}_\ell(2))^{G_{\mathbb{F}_q}}$.

Since Y is the blowup of X along a smooth cubic surface $S \subset X$, and S is geometrically rational, we similarly find that $\mathrm{CH}^2(Y) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \cong H^4(\overline{Y}, \mathbb{Q}_\ell(2))^{G_{\mathbb{F}_q}}$.

Let V be a singular fiber of f over a closed point $x \in \mathbb{P}^1_{\mathbb{F}_q}$. By standard transfer arguments we may reduce to the case where x is $\mathbb{F}_q$-rational. Then V is a nodal cubic threefold over $\mathbb{F}_q$, which is birational to $\mathbb{P}^3_{\mathbb{F}_q}$ via projection from the node. Using this, the Chow groups and homology of V are straightforward to control. From here we quickly obtain similar control over the invariants of the union Z of all singular fibers of f.

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