ON JACOBIANS OF HEISENBERG CURVES

DIMITRIOS NOULAS
NATIONAL AND KAPODISTRIAN UNIVERSITY OF ATHENS, GREECE
dnoulas@math.uoa.gr, noulasd.github.io

THEOREM (2026) [5]

Fix an odd prime ℓ. Let X_{ℓ^n} be a non-singular projective curve over $\mathbb{C}$ (or $\overline{\mathbb{Q}}$) such that:

$$X_{\ell^n} \longrightarrow \mathbb{P}^1_{\mathbb{C}} \text{ is a morphism of curves, branched over } 0,1,\infty, \text{ with}$$

$$\mathrm{Gal}(X_{\ell^n}/\mathbb{P}^1) \cong H_{\ell^n} := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, * \in \mathbb{Z}/\ell^n\mathbb{Z} \right\},$$

the discrete Heisenberg group modulo ℓ^n. Then, for the Jacobian variety $\mathrm{Jac}(X_{\ell^n})$ we have that:

$$\mathbf{Jac(X_{\ell^n}) \text{ is not of Complex Multiplication type} \iff \ell^n \neq 3}.$$

QUESTION (YASUTAKA IHARA)

Y. Ihara and G.W. Anderson in [1] completely determined the fixed field K_ϕ of the kernel of the pro-ℓ Galois representation:

$$\phi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Out}\left(\pi_1^{\text{ét}}(\ell)\left(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}\right)\right)$$

Prior to that, Ihara was trying to prove $K_\phi/\mathbb{Q}(\mu_{\ell^\infty})$ is a non-abelian extension, where μ_{ℓ^∞} is the group of all ℓ-power roots of unity. He tried using "towers" of curves and the fact that $G_{\mathbb{Q}}$ acts in an abelian way on the Tate modules when the Jacobians have CM. In 1987, Ihara [2] wrote:

"For $\ell > 3$ one might use Heisenberg curves instead, but at present, the author does not know whether their Jacobians do not really have enough CM"

QUESTION (FRANS OORT)

Curves $\mathcal{C}$ of genus $g \geq 2$ are said to *have many automorphisms* if their corresponding point at the Moduli space M_g is a "local maximum" in terms of the size of $\mathrm{Aut}(\mathcal{C})$.

If $\mathcal{C}$ has many automorphisms, does this imply $\mathrm{Jac}(\mathcal{C})$ is of CM-type?

No. But only *finitely* many counterexamples were known. Mostly some "exceptional" hyperelliptic and superelliptic curves (with bounded genus 1830):

$$y^2 = f(x), \ y^m = f(x).$$

Wolfart's criterion: Curves that yield normal topological covers of $\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$ have m.a.. Since $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\})$ has H_{ℓ^n} as a quotient, *Heisenberg curves have many automorphisms*. And their genus grows *unbounded*.

SKETCH OF PROOF ($\ell > 3$)

$X_{\ell^m} \to X_{\ell^n} \to \mathbb{P}^1$, is an inverse system of morphisms of curves. Thus, by standard theory of abelian varieties, the result "lifts" to the tower if it is shown for the base curves X_ℓ. Denote also by Y_ℓ the Fermat curve of exponent ℓ.

Via Chevalley-Weil understand the space

$$H^0(X_\ell, \Omega_{\mathbb{C}})$$

$$\cong H^0(Y_\ell, \Omega_{\mathbb{C}}) \times \prod_{j=1}^{\ell-1} V_j$$

in terms of *isotypic* components as a $\mathbb{C}[H_\ell]$-module. (Or $\overline{\mathbb{Q}}[H_\ell]$ accordingly.)

By Wolfart's framework pass to an isogeny decomposition:

$$\mathrm{Jac}(X_\ell)$$

$$\sim \mathrm{Jac}(Y_\ell) \times \prod_{j=1}^{\ell-1} \mathcal{A}_j^{k_j},$$

where $\mathcal{A}_j$ are simple abelian varieties over $\mathbb{C}$.

- Classification of endomorphism algebras of simple abelian varieties over $\mathbb{C}$ due to Albert and Shimura.
- V_j admits a cyclic $\mathrm{Gal}(X_\ell/Y_\ell)$ action, which yields $\mathbb{Q}(\zeta_\ell) \subset D_j$ for the endomorphism algebra D_j of each $\mathcal{A}_j$. Thus, $\dim D_j \geq \ell - 1$ over $\mathbb{Q}$.
- **Assume** CM-type for each $\mathcal{A}_j$. That is D_j is a CM-field (dimension $q_j^2 = 1$ over its center), with $2\dim_{\mathbb{C}} \mathcal{A}_j = \dim_{\mathbb{Q}} D_j$.
- $\ell \leq k_j q_j$ by Wolfart's framework and the above dots. Also $\dim_{\mathbb{C}} V_j = \dim_{\mathbb{C}} \mathcal{A}_j^{k_j} = \ell(\ell-3)/2$.

Combine the dots to arrive at $\frac{(\ell-1)}{2} \leq \dim_{\mathbb{C}} A_j \leq \frac{(\ell-3)}{2}$, **a contradiction!**

AFFINE MODELS

Let $\overline{\mathbb{Q}}(t)$ correspond to the projective t-line. Let $x = t^{1/\ell^n}, y = (1-t)^{1/\ell^n}$. Then $x^{\ell^n} + y^{\ell^n} = 1$. Let ζ be a primitive ℓ^n-th root of 1. Get a cyclic extension of $\overline{\mathbb{Q}}(x,y)$ by

$$\varepsilon^{\ell^n} = \prod_{i=1}^{\ell^n-1} (1 - \zeta^i x)^i.$$

REFERENCES

[1] G. Anderson and Y. Ihara. *Pro-ℓ branched coverings of $\mathbb{P}^1$ and higher circular ℓ-units.* Ann. of Math. (2), 128(2):271-293, 1988.

[2] Y. Ihara. *Some problems on three-point ramifications and associated large Galois representations.* In *Galois representations and arithmetic algebraic geometry (Kyoto, 1985/Tokyo, 1986)*, volume 12 of *Adv. Stud. Pure Math.*, pages 173–188. North-Holland, Amsterdam, 1987.

[3] J. Wolfart. *Triangle groups and Jacobians of CM type.* Available at: www.math.uni-frankfurt.de/ wolfart/Artikel/jac.pdf.

[4] F. Oort. *CM Jacobians.* Available on his website.

[5] D. Noulas. *On periods and Jacobians of Heisenberg curves.* Preprint, *arxiv.org/abs/2606.07315*, 2026.