Computing genus 2 curves whose Jacobians have good reduction away from 2

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Introduction

Let K be a number field, d a positive integer, and S a finite set of primes of K. In the 1960s, Shafarevich [5] conjectured that there exist only finitely many principally polarised abelian varieties A/K of dimension d with good reduction outside S. This was eventually proven by Gerd Faltings [2] in 1983, considered one of the crowning achievements of 20th century arithmetic geometry!

In particular, for any positive integer $g \geq 2$, this implies that there are only finitely many smooth genus g curves C/K whose Jacobians have good reduction outside S, as a genus g curve C/K is uniquely determined by its Jacobian (as a principally polarised abelian variety).

Unfortunately, Faltings's proof doesn't give an effective algorithm to compute all such dimension d abelian varieties A/K with good reduction outside S. While many effective algorithms have been developed in the case $d = 1$, effectively solving this problem in higher dimensions remains a hard challenge. Developing an algorithm that can effectively output a set of all such dimension d abelian varieties A/K with good reduction away from S is known as *the effective Shafarevich problem*, which remains unsolved.

Our work attempts to make some progress on the effective Shafarevich problem for $K = \mathbb{Q}$, $d = 2$, and $S = \{2\}$.

Poonen's question

In the 1990s, Merriman-Smart [3] and Smart [4] computed all $\mathbf{366}$ genus 2 curves $C/\mathbb{Q}$ with good reduction away from 2. In 1996, at ANTS II, Bjorn Poonen then proposed the following more general problem:

List all genus 2 curves $C/\mathbb{Q}$ whose Jacobians have good reduction away from 2.

Unfortunately, this is a strictly harder problem, as one can find many genus 2 curves $C/\mathbb{Q}$ where its Jacobian has good reduction at some odd prime p, but the curve $C/\mathbb{Q}$ itself has bad reduction at p. Some examples include:

- $C : y^2 = x^5 - 14x^3 + 81x$ has bad reduction at $\{2, 3\}$.
- $C : y^2 = 2x^5 - 9x^4 - 24x^3 + 22x^2 + 78x - 41$ has bad reduction at $\{2, 5\}$.
- $C : y^2 = 2x^5 + x^4 - 16x^3 - 72x^2 + 240x + 136$ has bad reduction at $\{2, 7\}$.
- $C : y^2 = x^5 + 478x^3 + 57122x$ has bad reduction at $\{2, 13\}$.
- $C : y^2 = 42x^6 + 1638x^5 + 189x^4 - 3696x^3 + 756x^2 + 1512x - 1428$ has bad reduction at $\{2, 3, 7\}$.
- $C : y^2 = x^5 + 28x^4 - 868x^3 - 6160x^2 + 43076x - 149072$ has bad reduction at $\{2, 3, 11\}$.

A provably complete classification of all such curves still remains out of reach. We instead combine various strategies to assemble a partial list of curves, extending Merriman–Smart and Smart's classical list of 366 curves.

Main results

There are at least $\mathbf{234}$ $\mathbb{Q}$-isogeny classes of abelian surfaces $A/\mathbb{Q}$ with good reduction away from 2. In particular, there are at least $\mathbf{512}$ $\mathbb{Q}$-isomorphism classes of genus 2 curves $C/\mathbb{Q}$ whose Jacobian has good reduction away from 2 (lying in $\mathbf{175}$ $\mathbb{Q}$-isogeny classes).

This adds $\mathbf{146}$ new curves to Smart's list of 366, each with bad reduction at one or more odd primes.

Our list includes all such curves with good reduction away from $\{2, 3\}$, $\{2, 5\}$, or $\{2, 7\}$. In particular there are exactly 78, 28, 24 such curves with $\mathrm{rad}(\Delta_{\min}) = 6$, 10, 14 respectively, and every such curve with $|\Delta_{\min}| \leq 10^{14}$ lies in our list. A list of all curves found can be found at:

`https://rvisser7.github.io/genus2/` .

A simpler problem

One strategy is to consider the simpler problem of computing genus 2 curves $C/\mathbb{Q}$ whose Jacobian is good outside 2, and where the curve $C/\mathbb{Q}$ itself has good reduction outside some given finite set of primes S.

Let $C : y^2 = c\,(x-\alpha_1)\cdots(x-\alpha_6)$ be good outside S with $\mathrm{Jac}(C)$ good away from 2. The Weierstrass points α_i generate the 2-torsion field $\mathbb{Q}(J[2])$, which is **unramified outside** 2. The problem essentially breaks down into two main steps:

1. Classify the 2-torsion fields $\mathbb{Q}(J[2])$

By Merriman–Smart [3], every number field of degree ≤ 6 unramified outside 2 has degree 1, 2, or 4 — and there are only 11 of them. In particular, $\mathbb{Q}(J[2])$ lies in one of the following three octic fields:

$$M_1 = \mathbb{Q}(\zeta_{16}), \quad M_2 = \mathbb{Q}(\zeta_8, \sqrt[4]{2}), \quad M_3 = \mathbb{Q}(\sqrt[4]{2\sqrt{2}-3}).$$

2. Compute $\alpha_i - \alpha_j$ by solving unit equations

Let $L = \mathbb{Q}(J[2])$ and let T be the set of primes in L lying above S. Siegel's identity gives a **T-unit equation** on the cross-ratios:

$$\frac{(\alpha_i-\alpha_j)(\alpha_k-\alpha_\ell)}{(\alpha_i-\alpha_k)(\alpha_j-\alpha_\ell)} + \frac{(\alpha_i-\alpha_\ell)(\alpha_j-\alpha_k)}{(\alpha_i-\alpha_k)(\alpha_j-\alpha_\ell)} = 1,$$

i.e. $x+y=1$ with $x, y \in \mathcal{O}_{L,T}^{\times}$. These can be effectively solved using Baker's bounds on linear forms in logarithms [1].

Further optimisations

To more efficiently solve for the roots, α_i, we fix a T-unit basis $\psi_1, \ldots, \psi_t$ of $\mathcal{O}_{L,T}^{\times}$ and write

$$\alpha_i - \alpha_j = \psi_1^{a_{1,i,j}} \psi_2^{a_{2,i,j}} \cdots \psi_t^{a_{t,i,j}}, \qquad a_{k,i,j} \in \mathbb{Z}.$$

Recovering the curve amounts to solving for the integer exponents $a_{k,i,j}$. These satisfy a large system of **linear constraints**, which include:

- **Galois constraints:** $a_{k,i,j} = a_{\sigma(k),\sigma(i),\sigma(j)}$ for all $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$.
- **Solutions to unit equations:** each solved cross-ratio pins down a linear relation among the $a_{k,i,j}$.
- **Cluster pictures:** good reduction of $\mathrm{Jac}(C)$ at odd primes p forces equalities among the p-adic valuations $v_p(\alpha_i - \alpha_j)$.

By results of Birch–Merriman and Evertse–Győry, the exponents $a_{k,i,j}$ can be effectively bounded, after which all integer solutions can be enumerated via either the closest-vector problem or integer linear programming.

Update (8 June 2026)

Recently, Raymond van Bommel, Céline Maistret, Jia Shi, and Andrew V. Sutherland have found $\mathbf{10}$ new genus 2 curves $C/\mathbb{Q}$ whose Jacobians have good reduction away from 2. This brings the total number of known such genus 2 curves $C/\mathbb{Q}$ to $\mathbf{522}$. I highly recommend reading their paper, available at:

`https://arxiv.org/abs/2606.09512` .

References

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