p-adic cohomologies and explicit arithmetic

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Any mistakes are the fault of the scribe

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1.2 What are periods?

1.2.1 Kontsevich-Zagier periods

Definition 1.2.1.1 (Kontsevich-Zagier)**.** An effective complex period is a number such that both its real and imaginary parts are absolutely convergent integrals of the form $\int_D f$, where

1. $D \subseteq \mathbb{R}^n$ us a semi-algebraic set (that is, it is obtained by union, intersection, and complement from subsets defined by polynomial inequalities with rational coefficients);
2. f is a rational function in $\mathbb{Q}(x_1, \dots, x_n)$.

We denote the set of such periods by $\mathcal{P}_{\mathrm{KZ}}^{\mathrm{eff}}$. We further set $\mathcal{P}_{\mathrm{KZ}} := \mathcal{P}_{\mathrm{KZ}}^{\mathrm{eff}}[\frac{1}{2\pi i}]$.

Example 1.2.1.2. The following numbers are (Kontsevich-Zagier) periods.

1. Let $a \in \mathbb{Q}\times$. We have
$$\int_{1 \leq x \leq a} \frac{dx}{x} = \log(a) \in \mathcal{P}_{\mathrm{KZ}}$$
2. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \pi = \int_{x^2+y^2 \leq 1} dx\, dy$
3. Given $a \in \mathbb{Q}$, we have $\int_{4x^2 \leq a} dx = \sqrt{a} \in \mathcal{P}_{\mathrm{KZ}}$.
4. $\zeta(2) = \sum_{n \geq 1} \frac{1}{n^2} = \int_{0<x<y<1} \frac{dx\, dy}{(1-x)y}$.
5. Elliptic integrals: $\int_1^2 \frac{dt}{t^3+1} = \int_D \frac{dt}{s}$, where $D = \{(s,t) \bigm| s^2 = t^3+1, s \geq 0, 1 \leq t \leq 2\}$.

   Note of the scribe: the domain D has measure 0, so this doesn't quite fit the definition as given. I believe one can integrate over a 2-dimensional domain with suitable inequalities and get the same period.

Exercise 1.2.1.3. *Every $\alpha \in \overline{\mathbb{Q}}$ is in $\mathcal{P}_{\mathrm{KZ}}$.*

Remark 1.2.1.4. 1. $\mathcal{P}_{\mathrm{KZ}}$ is as subring of $\mathbb{C}$ containing $\overline{\mathbb{Q}}$.

2. Every $\alpha \in \mathcal{P}_{\mathrm{KZ}}$ has many representations $\int_D f$.

1.2.2 A second definition of periods

Definition 1.2.2.1. Let $X/\mathbb{Q}$ be a smooth algebraic variety of dimension d, $D \subset X$ a normal crossing divisor (defined over $\mathbb{Q}$), and $\omega \in H^0(X, \Omega_X^d)$, and γ a singular chain in $X(\mathbb{C})$ whose boundary $\partial\gamma$ belongs to $D(\mathbb{C})$. An *nc (normal crossing) period* is a number of the form $\int_\gamma \omega$.

We denote by $\mathcal{P}_{\mathrm{nc}}$ the subring of $\mathbb{C}$ generated by $\int_\gamma \omega$ for (X, D, ω, γ) as above and we invert $2\pi i$.

Example 1.2.2.2.
- Let $X = \mathbb{A}^1_{\mathbb{Q}}$, $\alpha \in \overline{\mathbb{Q}}$, $D = \{\text{conjugates of } \alpha\}$, $\omega = dz$, $\gamma : t \mapsto t\alpha$. Then $\int_\gamma \omega = \alpha$: thus, $\mathcal{P}_{\mathrm{nc}}$ contains $\overline{\mathbb{Q}}$.
- $X = \mathbb{G}_m$, $D = \{1, a\}$ with $a \in \mathbb{Q}^\times$, and γ a path from 1 to a. Then $\int_\gamma \omega = \log a$.
- $X = \mathbb{G}_m$, $D = \emptyset$, γ the radius 1 circle around 0, $\omega = \frac{dz}{z}$. Then we have $\int_\gamma \omega = 2\pi i$.

Remark 1.2.2.3. With both definitions, the ring of periods is countable: there are only countably many semi-algebraic sets and countably many rational functions, and similarly there are only countably many varieties over $\mathbb{Q}$, divisors defined over $\mathbb{Q}$, and differential forms.

Exercise 1.2.2.4. *If in the definition of nc periods we replace $\mathbb{Q}$ with any algebraic extension of $\mathbb{Q}$, we get the same ring of periods.*

1.2.3 Cohomological periods

Let $X/\mathbb{Q}$ be a smooth algebraic variety, $\Omega_X^\bullet$ the algebraic de Rham complex.

Definition 1.2.3.1. $H^\bullet_{\mathrm{dR}}(X/\mathbb{Q}) := \mathbb{H}^\bullet(X, \Omega_X^\bullet)$.

Example 1.2.3.2. 1. If X is affine, then $\mathbb{H}^\bullet = H^\bullet$ and

$$H^\bullet_{\mathrm{dR}}(X/\mathbb{Q}) = H^\bullet(\mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots).$$

2. If $X = \mathbb{G}_m$, $\Omega^1 = \mathbb{Q}[t^{\pm 1}]\,dt$. We consider

$$\mathbb{Q}[t^{\pm 1}] \xrightarrow{d} \mathbb{Q}[t^{\pm 1}]\,dt;$$

the image of d is $\langle t^n dt \mid n \neq -1\rangle$. It follows from this that

$$H^\bullet_{\mathrm{dR}}(\mathbb{G}_m/\mathbb{Q}) = \mathbb{Q}[0] \oplus \mathbb{Q}\frac{dt}{t}[-1].$$

Exercise 1.2.3.3. *Let $P(x) \in \mathbb{Q}[x]$ be a degree 3 polynomial with simple roots. Let $E : y^2 = P(x)$ in $\mathbb{A}^2_{\mathbb{Q}}$ and let $R := \frac{\mathbb{Q}[X,y]}{(P)}$. Show that $\Omega^1_E = \frac{\mathbb{Q}[x,y]}{(P)}\{dx, dy\}$. Consider*

$$R \xrightarrow{d} \Omega^1_{R/\mathbb{Q}}.$$

Show that $\omega := \frac{dx}{y}$ and $x\frac{dx}{y}$ form a basis of $\Omega^1_{R/\mathbb{Q}}$.

Hint. Since $P(x), P'(x)$ are coprime, use Bézout to write $AP + BP' = 1$. Then note $\omega = Ay\,dx + 2B\,dy$ (which shows that $\omega \in \Omega^1_{R/\mathbb{Q}}$).

Remark 1.2.3.4. We have $\Omega^\bullet_{X/\mathbb{Q}} \otimes_{\mathbb{Q}} K = \Omega^\bullet_{X_K/K}$.

Theorem 1.2.3.5 (Grothendieck)**.** *Let X be a smooth algebraic variety over $\mathbb{C}$. The natural map of complexes $\Omega^\bullet_{X/\mathbb{C}} \to \Omega^\bullet_{X(\mathbb{C}),\mathcal{C}^\infty}$ induces an isomorphism*

$$H^\bullet_{\mathrm{dR}}(X/\mathbb{C}) \cong H^\bullet_{\mathrm{dR},\mathcal{C}^\infty}(X(\mathbb{C}), \mathbb{C}).$$

Here, $H^\bullet_{\mathrm{dR},\mathcal{C}^\infty}$ is the 'usual' de Rham cohomology computed by smooth differential forms.

Remark 1.2.3.6. As a consequence of the theorem, if X is defined over $\mathbb{Q}$ we have

$$H^\bullet_{\mathrm{dR}}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^\bullet_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C}) \cong H^\bullet_{\mathrm{dR},\mathcal{C}^\infty}(X(\mathbb{C}), \mathbb{C}),$$

so that the (apparently more complicated) arithmetic object $H^\bullet_{\mathrm{dR}}(X/\mathbb{Q})$ is computed by something that is essentially topological.

Theorem 1.2.3.7 (Poincaré's lemma + de Rham's theorem)**.** *Let X be a differentiable manifold. There is a canonical isomorphism*

$$\begin{array}{rccc} \mathrm{comp}_{\mathrm{dR},B}(X): & H^\bullet_{\mathrm{dR},\mathcal{C}^\infty}(X,\mathbb{C}) & \xrightarrow{\sim} & H^\bullet_{\mathrm{sing}}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \\ & \omega & \mapsto & \left(\gamma \mapsto \int_\gamma \omega\right). \end{array}$$

On the right, using the universal coefficient theorem we identify $H^\bullet_{\mathrm{sing}}(X,\mathbb{Q})$ with $H_{\mathrm{sing},\bullet}(X,\mathbb{Q})^\vee$, the dual of topological homology.

Corollary 1.2.3.8. *Let $X/\mathbb{Q}$ be a smooth algebraic variety. There is a canonical isomorphism*

$$\begin{array}{ccc} H^\bullet_{\mathrm{dR}}(X/\mathbb{Q}) \otimes \mathbb{C} & \to & H^\bullet_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{Q}) \otimes \mathbb{C} \\ \omega & \mapsto & \left(\gamma \mapsto \int_\gamma \omega\right). \end{array}$$

Definition 1.2.3.9 (Cohomological periods)**.** We let $\mathcal{P}_C$ be the subring of $\mathbb{C}$ generated by $\mathbb{Q}$, powers of $2\pi i$, and the matrix coefficients of $\mathrm{comp}_{dR,B}(X)$, for all smooth algebraic varieties $X/\mathbb{Q}$ and all choices of bases for $H^\bullet_{\mathrm{dR}}(X/\mathbb{Q})$ and $H^\bullet_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{Q})$.

Remark 1.2.3.10. Here we consider all differential forms, while in the nc definition we only work with $H^0(X,\Omega^d_X)$, which is just one piece of the Hodge filtration. Also, the nc definition uses relative cohomology, while here we don't do this (but we could); one gets the same ring of periods.

Theorem 1.2.3.11 (Huber-Müller–Stach; formerly a conjecture of Kontsevich)**.** *The equalities $\mathcal{P}_{\mathrm{KZ}} = \mathcal{P}_{\mathrm{nc}} = \mathcal{P}_c$ holds.*

Exercise 1.2.3.12. *Recall $\zeta(2) = \int_{0<x<y<1} \frac{dx\,dy}{(x-1)y}$. The differential form we're integrating has poles at the boundary (along $x = 1, y = 0$). Exhibit $\zeta(2)$ in $\mathcal{P}_{\mathrm{nc}}$ and $\mathcal{P}_c$.*

Hint. Blow up at the singularities.

Conjecture 1.2.3.13 (Kontsevich-Zagier, vague formulation)**.** *All equalities between periods are justified by additivity of integrals, the change of variables formula, and Stokes' theorem.*

Remark 1.2.3.14. To formulate this conjecture precisely, one introduces a ring of formal periods, built out of the formal data of periods and quotiented out by formal additivity, formal change of variables formula, and formal Stokes; there is an evaluation map from formal periods to actual periods, and the conjecture is that this map is injective.

1.3 Grothendieck's period conjecture

Let X be a smooth proper variety of dimension d, $Z \subset X$ be smooth of codimension c. There is a topological cycle class map (given by triangulation)

$$Z \longrightarrow [Z] \in H^{\mathrm{sing}}_{2d}(X(\mathbb{C}),\mathbb{Q}) \overset{\text{Poincaré}}{\cong} H^{2c}_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{Q}),$$

and a de Rham cycle class map

$$Z \longrightarrow [Z] \in H^{\mathrm{sing}}_{2d}(X(\mathbb{C}),\mathbb{Q}) \cong H^{2d}_{\mathrm{dR}}(X/\mathbb{Q})^\vee \overset{\text{Poincaré}}{\cong} H^{2c}_{\mathrm{dR}}(X/\mathbb{Q}).$$

These two cycle class maps will be denoted $\mathrm{cl}_B(Z), \mathrm{cl}_{\mathrm{dR}}(Z)$.

Tensoring up with $\mathbb{C}$, we have the comparison map $\mathrm{comp}_{\mathrm{dR},B}$, which satisfies

$$\mathrm{comp}_{\mathrm{dR},B}(\mathrm{cl}_{\mathrm{dR}}(Z)) = \mathrm{cl}_B(Z)$$

up to powers of $2\pi i$.

Remark 1.3.0.1. We're being a bit cavalier with Tate twists.

Note of the scribe. I think the missing factor is $(2\pi i)^c$.

Conjecture 1.3.0.2 (Weak Grothendieck period conjecture, wGPC). *The inclusion*

$$\mathcal{Z}^c(X)_{\mathbb{Q}} \xrightarrow{\text{cycle class maps}} H^{2c}_{\mathrm{dR}}(X/\mathbb{Q}) \cap H^{2c}_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$$

is surjective, where $\mathcal{Z}^c(X)_{\mathbb{Q}}$ is the group of algebraic cycles of X of codimension c (with $\mathbb{Q}$-coefficients).

For c odd, the intersection $H^c_{\mathrm{dR}}(X/\mathbb{Q}) \cap H^c_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$ is trivial.

Remark 1.3.0.3. Status of the conjecture: wGPC holds for $c = 1$ and for abelian varieties, curves, K3 surfaces, some HyperKähler varieties... This is due to Bost-Charles. The crucial ingredient is the Wüstholz analytic subgroup theorem.

Remark 1.3.0.4 (Chowla-Selberg formula). Consider the elliptic curve $E : y^2 = 4x^3 + 4x$, with CM by $\mathbb{Z}[i]$. The periods of E are (generated by)

$$\alpha := \int_{\gamma_1} \frac{dx}{y} = \frac{\Gamma(1/4)^2}{2^{3/2}\sqrt{\pi}}, \quad i\alpha, \quad -\frac{\pi}{\alpha}, \quad -i\frac{\pi}{\alpha}.$$

Grothendieck's period conjecture implies that $\alpha, \pi/\alpha$ are irrational.

Chapter 2

Periods (complex and p-adic) and applications II

2.1 Complements on de Rham cohomology

Let X be a separated scheme, $\mathcal{F}$ be a quasi-coherent sheaf on X. If X is affine, then $H^i(X, \mathcal{F}) = (0)$ for all $i > 0$. When all higher cohomology of a sheaf $\mathcal{F}$ vanishes, we say that $\mathcal{F}$ is *acyclic*. Let $j : U \hookrightarrow X$ be an affine open and let $\mathcal{F}$ be a quasi-coherent sheaf on U. Then $j_*\mathcal{F}$ is acyclic on X (that is, $H^i(X, j_*\mathcal{F}) = (0)$ for all $i > 0$).

Now let $X = U_1 \cup U_2$, with each U_i open and affine, and let $j_k : U_k \hookrightarrow X$ be the inclusion, for $k = 1, 2$. There is an exact sequence

$$0 \to \mathcal{F} \to j_{1,*}\mathcal{F}|_{U_1} \oplus j_{2,*}\mathcal{F}|_{U_2} \to j_{12,*}\mathcal{F}|_{U_1 \cap U_2} \to 0.$$

Since U_1, U_2 and $U_1 \cap U_2$ are affine (because X is separated, so the intersection of affine subschemes is affine), the second and third term of the above complex are acyclic, and so we've found a resolution of $\mathcal{F}$ by acyclic sheave, which may be used to compute the cohomology of $\mathcal{F}$.

This idea can be extended to define hypercohomology. Let $\mathcal{F}_\bullet \in D^b(\mathrm{QCoh}(X))$ be a complex of sheaves. For each $\mathcal{F}_i$, we can find an acyclic resolution $\mathcal{F}_i \to A_i^0 \to A_i^1 \to \cdots$ of $\mathcal{F}_i$. Consider the double complex

$$\begin{array}{ccccccc}
\mathcal{F}_0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A_0^0 & \longrightarrow & A_1^0 & \longrightarrow & A_2^0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A_0^1 & \longrightarrow & A_1^1 & \longrightarrow & A_2^1 & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

Since the terms A_i^j are acyclic, there is a quasi-isomorphic to the totalisation $\mathrm{Tot}\,(\mathcal{A}_\bullet^\bullet, d_h \pm d_v)$. By definition, $\mathbb{H}^\bullet(X, \mathcal{F}^\bullet) = H^\bullet(H^0(\mathrm{Tot}\,(\mathcal{A}_\bullet^\bullet, d_h \pm d_v)))$.

Example 2.1.0.1 (Cohomology of $\mathbb{P}^1$). Let $X = \mathbb{P}^1$ and consider the complex $\mathcal{O}_{\mathbb{P}^1} \to \Omega^1$. Write $U_1 = \mathrm{Spec}\, K[t], U_2 = \mathrm{Spec}\, K[t^{-1}]$. To get acyclic resolutions, we consider as above

$$0 \to \mathcal{F} \to j_{1,*}\mathcal{F}|_{U_1} \oplus j_{2,*}\mathcal{F}|_{U_2} \to j_{12,*}\mathcal{F}|_{U_1 \cap U_2} \to 0.$$

We get a diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^1} \arrow[r] \arrow[d] & \Omega^1_{\mathbb{P}^1} \arrow[d] \\
j_{1,*}\mathcal{O}_{U_1} \oplus j_{2,*}\mathcal{O}_{U_2} \arrow[r] \arrow[d] & j_{1,*}\Omega^1_{U_1} \oplus j_{2,*}\Omega^1_{U_2} \arrow[d] \\
j_{12,*}\mathcal{O}_{U_1\cap U_2} \arrow[r] \arrow[d] & j_{12,*}\Omega^1_{U_1\cap U_2} \arrow[d] \\
0 & 0
\end{tikzcd}$$

Hence,

$$\mathbb{H}^\bullet(\mathbb{P}^1, \Omega^\bullet_{\mathbb{P}^1}) = H^\bullet\left(H^0(\mathcal{O}_{U_1}) \oplus H^0(\mathcal{O}_{U_2}) \to H^0(\mathcal{O}_{U_1\cap U_2}) \oplus H^0(\Omega^1_{U_1}) \oplus H^0(\Omega^1_{U_2}) \to H^0(\Omega^1_{U_1\cap U_2})\right).$$

The maps are (up to sign)

$$(f,g) \mapsto (f - g|_{U_1\cap U_2},\ df,\ dg)$$

and

$$(f,\omega_1,\omega_2) \mapsto df - \omega_1|_{U_1\cap U_2} + \omega_2|_{U_1\cap U_2}.$$

This gives the explicit complex of vector spaces

$$0 \to k[t] \oplus k[t^{-1}] \to k[t^{\pm 1}] \oplus k[t]\,dt \to k[t^{\pm 1}]\,dt \to 0,$$

of which we need to compute the cohomology. One finds $H^\bullet_{\mathrm{dR}}(\mathbb{P}^1/\mathbb{Q}) = \mathbb{Q}[0] \oplus \mathbb{Q}\frac{dt}{t}[-2]$.

Exercise 2.1.0.2. *1. Finish this computation.*

2. Compute $H^\bullet_{\mathrm{dR}}(\mathbb{A}^1 \setminus$ finitely many points).

3. Let $E : y^2z = x^3 + axz^2 + bz^3$ in $\mathbb{P}^2$. Let U, V be the open subschemes $E\setminus\{\infty\}$ and $V = E\setminus\{y = 0\}$. Compute $H^\bullet_{\mathrm{dR}}(E/K)$.

Hint. $H^2_{\mathrm{dR}}(E/K)$ is spanned by $\frac{x^2}{y^2}\,dx$

Remark 2.1.0.3. Computing de Rham cohomology for affine varieties is relatively easy. However, proving the comparison theorem between algebraic and differential de Rham cohomology seems to require going through (smooth projective) varieties, even if one is only interested in the affine ones, because the proof ultimately reduces to the GAGA principle.

2.2 Grothendieck's period conjecture

Let X be a smooth projective over $\mathbb{Q}$, or even $\overline{\mathbb{Q}}$. We have a comparison isomorphism

$$\mathrm{comp}_{\mathrm{dR},B}(X) : H^\bullet_{\mathrm{dR}}(X/\mathbb{Q}) \otimes \mathbb{C} \to H^\bullet_B(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C}.$$

Choosing rational bases we get $\Omega_X \in \mathrm{GL}_N(\mathbb{C})$, for $N = \dim_{\mathbb{Q}} H^\bullet_{\mathrm{dR}}(X)$. It turns out that an algebraic cycle α on X^n produces a polynomial function P_α on $\mathrm{GL}_{N,\mathbb{Q}}$ such that $P_\alpha(\Omega_X)$.

Conjecture 2.2.0.1 (Grothendieck period conjecture, preliminary version)**.** *We have*

$$\{P \in \mathcal{O}(\mathrm{GL}_{N,\mathbb{Q}}) \mid P(\Omega_X) = 0\} = \{P_\alpha \mid \alpha \text{ algebraic cycle on } X^n \text{ for some } n \geq 1\}.$$

In other words, the ideal of $\overline{\Omega_X}^{\mathrm{Zar}}$ in $\mathrm{GL}_{N,\mathbb{Q}}$ is given by the polynomials arising from algebraic cycles.

Remark 2.2.0.2. Status: widely open! It's known for $\mathbb{G}_m$, because it's essentially the transcendence of π, and for CM elliptic curves, by work of Chudnowski.

Remark 2.2.0.3. The dimension of $\overline{\Omega_X}^{\mathrm{Zar}}$ coincides with the transcendence degree $\mathrm{trdeg}_{\mathbb{Q}}(\mathbb{Q}(\text{coefficients of } \Omega_X))$.

2.3 Some Tannakian formalism (Galois-Poincaré)

Definition 2.3.0.1. A K-linear Tannakian category is a K-linear abelian category $\mathcal{C}$ endowed with a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

which is associative and symmetric and has a unit $\mathbf{1}$. All objects have duals, denoted by $X \rightsquigarrow X^\vee$, that come endowed with evaluation and coevaluation maps

$$\mathrm{ev} : X^\vee \otimes X \to \mathbf{1},$$

$$\mathrm{coev} : \mathbf{1} \to X \otimes X^\vee$$

such that

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\mathrm{id}"'] & X \otimes X^\vee \otimes X \arrow[d, "\mathrm{id} \otimes \mathrm{ev}"] \\ & X \end{tikzcd}$$

commutes. Finally, $\mathbf{1}$ is supposed to satisfy $\mathrm{End}(\mathbf{1}) = K$.

Example 2.3.0.2.
1. Finite-dimensional K-vector spaces
2. If Γ is a discrete group, the category of finite-dimensional representations of Γ
3. If G is an affine group scheme over K, the category of algebraic representations of G over K.
4. If X is a topological space, the category of (finite dimensional) local systems $\mathrm{LocSys}^{\mathrm{f.g.}}(X, K)$.
5. $\mathrm{Vect}_K^{\mathbb{Z}}$, the $\mathbb{Z}$-graded vector spaces.
6. sVect_K, the category of super vector spaces ($\mathbb{Z}/2\mathbb{Z}$-graded vector spaces over K), with symmetry map

$$\begin{array}{ccc} V \otimes W & \to & W \otimes V \\ v \otimes w & \mapsto & (-1)^{|v|\cdot|w|} w \otimes v. \end{array}$$

Definition 2.3.0.3. Let L be a field extension of K. A *fibre functor* on a Tannakian category $\mathcal{C}$ is a functor

$$F : \mathcal{C} \to \mathrm{Vect}_L$$

which is exact, faithful, and symmetric monoidal. The last condition means that F comes equipped with isomorphisms $F(X) \otimes F(Y) \xrightarrow{\theta_{X,Y}} F(X \otimes Y)$ and $K \xrightarrow{\eta_1} F(\mathbf{1})$ with the obvious compatibilities.

The category $\mathcal{C}$ is said to be *neutral* if it admits a fibre functor to Vec_K.

Example 2.3.0.4. In our previous examples, (1), (2) and (4) are neutral, with fibre functor the obvious forgetful functor. Example (3) is also neutral, with fibre functor given by taking the fibre at a point $x \in X$. Finally, example (5) is not neutral.

Theorem 2.3.0.5 (Saavedra-Rivano, Deligne)**.** *Let $\mathcal{C}, F : \mathcal{C} \to \mathrm{Vec}_K$ be a neutral Tannakian category. The functor $\mathrm{Aut}^\otimes(F) : R \mapsto \mathrm{Aut}^\otimes(F(-) \otimes_K R)$ is representable by an affine group scheme G_F over K, and F factors uniquely as*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\sim"] \arrow[dr, "F"'] & \mathrm{Rep}_{G_F} \arrow[d, "\mathrm{forget}"] \\ & \mathrm{Vec}_K \end{tikzcd}$$

where the arrow $\mathcal{C} \cong \mathrm{Rep}_{G_F}$ is an equivalence of symmetric monoidal categories. Moreover, if $F' : \mathcal{C} \to \mathrm{Vec}_K$ is another fibre functor, then

$$\mathrm{Isom}^\otimes(F, F') : R \mapsto \mathrm{Isom}^\otimes(F(-) \otimes_K R, F'(-) \otimes_K R)$$

is also representable by an affine K-scheme, which is a G_F-torsor on the right and a $G_{F'}$-torsor on the left.

Exercise 2.3.0.6. *Prove the following.*

1. *Let* Γ *be a finite abstract group. Applying Tannakian reconstruction to* Rep_K^{Γ} *(with the obvious fibre functor) gives* $G_{\mathrm{forget}} = \mathrm{Spec}(\mathcal{O}(\Gamma))$.

2. *For* $\mathrm{Vec}_K^{\mathbb{Z}}$ *(with the forgetful functor), the Tannakian fundamental group is* $\mathbb{G}_m$ *(i.e., graded vector spaces are the same thing as representations of* $\mathbb{G}_m$*).*

3. $\Gamma = \mathbb{Z}$, $\mathrm{Rep}_{\mathbb{C}}(\mathbb{Z}) \to \mathrm{Vec}_{\mathbb{C}}$*: Tannakian reconstruction gives* $G_{\mathrm{Forget}} = \mathbb{G}_a \times D(\mathbb{C}^{\times})$*, where* $D(\mathbb{C}^{\times})$ *is the diagonalizable group with group of characters* $\mathbb{C}^{\times}$.

Chapter 3

Periods (complex and p-adic) and applications III

3.1 What have we done so far?

Let $X/\overline{\mathbb{Q}}$ be a smooth (projective) variety. We have considered the maps

$$\mathrm{comp}_{\mathrm{dR},B}(X) : H_{\mathrm{dR}}(X/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to H_B(X(\mathbb{C}), \overline{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$$

and

$$\mathrm{cl}_{\mathrm{dR},B} : Z(X)_{\overline{\mathbb{Q}}} \hookrightarrow H^\bullet_{\mathrm{dR}}(X/\overline{\mathbb{Q}}) \cap H^\bullet_B(X(\mathbb{C}), \overline{\mathbb{Q}})$$

The weak Grothendieck period conjecture is that the latter inclusion should be an equality. Note the similarity with the Hodge conjecture.

Let $\mathcal{P}_{\mathbb{C}}(X)$, the *algebra of complex periods of* X, be the $\mathbb{Q}$-bar subalgebra of $\mathbb{C}$ generated by the coefficients of $\mathrm{comp}_{\mathrm{dR},B}$ in $\overline{\mathbb{Q}}$-bases. We have seen examples showing that many interesting numbers (logarithms of rational numbers, multiple zeta values, $\Gamma\left(\frac{a}{d}\right)$, ...) are periods. From the formalism, we have a bound

$$\operatorname{trdeg} \mathcal{P}_{\mathbb{C}}(X) \leq \dim G_{\mathrm{dR}}.$$

The Grothendieck period conjecture predicts that this should be an inequality.

Goal for today. Construct a p-adic analogue $\mathcal{P}_p \subset \overline{\mathbb{Q}}_p$ that contains interesting numbers, in such a way that trdeg is controlled by algebraic classes via Tannakian groups. This is based on joint work of G. Ancona and D. Fratila.

3.2 Initial motivation

Let $X/\mathbb{F}_q$ be a smooth projective variety. The Tate conjecture predicts that the map

$$Z(X)_{\mathbb{Q}} \otimes \mathbb{Q}_\ell \hookrightarrow_{\mathrm{cl}} H^\bullet_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)^{\mathrm{Gal}_{\mathbb{F}_q}}$$

is surjective. Tate asked to understand the image of the map

$$Z(X)_{\mathbb{Q}} \hookrightarrow_{\mathrm{cl}} H^\bullet_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)^{\mathrm{Gal}_{\mathbb{F}_q}},$$

namely, understand the position of the actual algebraic cycles inside étale cohomology.

We consider $X/\mathbb{F}_p$ with a lift X_0 to $\mathbb{Q}$ (or more generally, $X/\mathbb{F}_q$ with a lift to a number field, or $X/\overline{\mathbb{F}}_p$ with a lift to $\overline{\mathbb{Q}}$).

In this case, we get a comparison

$$H^\bullet_{\mathrm{dR}}(X_0/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to H^\bullet_{\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell),$$

and this gives a rational subspace, but this isomorphism depends on the choice of inclusion of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_\ell$, and the periods one gets from this comparison depend on these choices. So we try something else.

3.3 Setting

Let $X_0/\mathbb{Q}$ have good reduction at p and denote the reduction by X_p. We have a comparison isomorphism (due to Berthelot)

$$\begin{tikzcd} Z(X_0) \arrow[r, "\mathrm{cl}_{dR}"] \arrow[d] & H_{\mathrm{dR}}(X_0/\mathbb{Q}) \otimes \mathbb{Q}_p \arrow[d] \\ Z(X_p) \arrow[r, "\mathrm{cl}_{cris}"'] & H_{cris}(X_p, \mathbb{Q}_p) \end{tikzcd}$$

We get

$$\Omega_p : Z(X_0) \hookrightarrow Z(X_p) \cap H_{\mathrm{dR}}(X_0/\mathbb{Q}).$$

Conjecture 3.3.0.1 (p-wGPC)**.** *Is this an equality?*

Example 3.3.0.2. As a special case, let X_0 be am abelian variety A with good reduction at p, denoted by A_p. Let $f \in \mathrm{End}^0(A_p)$ be an endomorphism. Then f acts on $H^1_{\mathrm{cris}}(A_p, \mathbb{Q}_p)$, which contains $H^1_{\mathrm{dR}}(A/\mathbb{Q})$. If f lifts to characteristic 0, this action will preserve $H^1_{\mathrm{dR}}(A/\mathbb{Q})$. The p-wGPC implies that f lifts to an endomorphism in characteristic 0 if and only if it preserves $H^1_{\mathrm{dR}}(A/\mathbb{Q})$.

Note that $f : A_p \to A_p$ gives a divisor $[\Gamma_f]$ in $Z^1(A \times A)$; it is this class that we apply the p-wGPC to.

Definition 3.3.0.3 (André's p-adic periods)**.** Let $\mathcal{P}_p^{\text{André}}(X)$ be the $\overline{\mathbb{Q}}$-subalgebra of $\overline{\mathbb{Q}}_p$ generated by the coefficients of

$$Z(X_0)_{\mathbb{Q}} \to H_{\mathrm{dR}}(X_0/\mathbb{Q}) \otimes \mathbb{Q}_p$$

with respect to $\mathbb{Q}$-bases. We further let $\mathcal{P}_p^{\text{André}}(\langle X_0\rangle)$ be the $\overline{\mathbb{Q}}$-subalgebra of $\overline{\mathbb{Q}}_p$ generated by $\mathcal{P}_p^{\text{André}}(X_p^n)$ for all $n \geq 1$.

Remark 3.3.0.4. There are other definitions of p-adic periods in the literature.

1. Fontaine's p-adic periods. Using the comparison isomorphism

   $$H_{\mathrm{dR}}(X_0/\overline{\mathbb{Q}})_{\otimes\overline{\mathbb{Q}}} B_{\mathrm{dR}} \xrightarrow{\sim} H_{\text{ét}}(X_0, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$

   one gets periods in B_{dR}. Let $\mathcal{P}_{\mathrm{Fon}}(X)$ be this ring of periods.

   Conjecture 3.3.0.5 (Fontaine). $\mathrm{trdeg}_{\overline{\mathbb{Q}}_p}(\mathcal{P}_{\mathrm{Fon}}) = \dim G_{\mathrm{dR}}(X_0)$.

   André proved that the transcendence degree on the left depends on the chosen embedding $\sigma : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, but also that the conjecture is true for 'generic' σ.

2. Brown-Furusho. Take $X_0/\mathbb{Q}$ with good reduction $X_p/\mathbb{F}_p$. There is an action of Fr^* on $H_{cris}(X_p/\mathbb{Q}_p)$, which contains $H_{\mathrm{dR}}(X/\mathbb{Q})$. Let $\mathcal{P}_{\mathrm{Frob}}(X)$ be the subalgebra of $\mathbb{Q}_p$ generated by the coefficients of Fr^* with respect to $H_{\mathrm{dR}}(X_0/\mathbb{Q})$.

   Notice that Fr_X^* is a $\mathbb{Q}_p$-point of $\mathrm{Aut}^{\otimes}(H_{\mathrm{dR}}|_{\langle X_0\rangle}) = G_{\mathrm{dR}}(X)(\mathbb{Q}_p)$. This implies $\mathrm{trdeg}\, \mathcal{P}_{\mathrm{Frob}}(X) \leq \dim G_{\mathrm{dR}}(X)$. However, the bound is **not** optimal, because the characteristic polynomial of Fr_X^* is in $\mathbb{Q}[T]$, and $G_{\mathrm{dR}}(X)$ does not see it. (More precisely, the conjugacy class of Frobenius admits at least one equation with rational coefficients, so Fr_X^* cannot be a generic point.)

   Note also that $\mathcal{P}_{\mathrm{Frob}}(X) \subset \mathcal{P}_p(X^2)$, because Γ_{Fr_X} is an element of $Z^1(X_p \times X_p)$.

3.4 Examples

3.4.1 $\mathbb{G}_m$

Let $X_0 = \mathbb{G}_m$. We compute

$$H^\bullet_{\mathrm{rig}}(X_p, \mathbb{Q}_p) = H^\bullet\left(A^\dagger_{\mathbb{Q}_p} \xrightarrow{d} \Omega^1_{A^\dagger_{\mathbb{Q}_p}/\mathbb{Q}_p}\right),$$

where $A^\dagger_{\mathbb{Q}_p}$ is the ring of overconvergent power series

$$A^\dagger_{\mathbb{Q}_p} = \left\{ \sum_{n\in\mathbb{Z}} a_n t^n \mid \lim_{|n|\to\infty} (v_p(a_n) - c(n)) = \infty \right\}.$$

One computes easily

$$H^\bullet_{\mathrm{rig}}(X_p, \mathbb{Q}_p) = \mathbb{Q}_p \oplus \mathbb{Q}_p \, d\log t.$$

There is again a comparison isomorphism

$$H_{\mathrm{dR}}(X_0/\mathbb{Q}) \otimes \mathbb{Q}_p \to H^\bullet_{\mathrm{rig}}(X_p, \mathbb{Q}_p),$$

where the map sends 1 to 1 and $d\log t$ to $d\log t$.

There is an action of $\mathrm{Fr}^*_{X_p}$ on $H^\bullet_{\mathrm{rig}}(\mathbb{G}_{m,\mathbb{F}_p}, \mathbb{Q}_p)$. How do we compute this action? Lift Fr_{X_p} to $A^\dagger$ (in the obvious way, $t \mapsto t^p$). Then apply this lift to $1, d\log t$; the pullback of $d\log t$ is $d\log(t^p) = p\, d\log t$. Thus, the matrix of Frobenius is $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$.

3.4.2 Kummer motive

This has to do with the relative cohomology of $\mathbb{G}_m$ with respect to two points $1, a$. Let $a \in \mathbb{Z}$, with $a \not\equiv 0, 1, -1 \pmod p$. There is a motive

$$\mathcal{K}_a := h^1(\mathbb{G}_m, \{1, a\}).$$

The de Rham realisation is

$$R_{\mathrm{dR}}(\mathcal{K}_a) = H^1_{\mathrm{dR}}(\mathbb{G}_m, \{1, a\});$$

the rigid realisation is

$$R_{\mathrm{rig}}\left((\mathcal{K}_a)_p\right) = H^1_{\mathrm{rig}}(\mathbb{G}_{m,\mathbb{F}_p}, \{1, a\}, \mathbb{Q}_p).$$

There is a non-split exact sequence

$$\mathbf{1} \to \mathcal{K}_a \to \mathbf{1}(-1)$$

in $\mathrm{Mot}(\mathbb{Q})$. Fact: $(\mathcal{K}_a)_p = \mathbf{1} \oplus \mathbf{1}(-1)$. Dualising, we get a non-split sequence

$$\mathbf{1}(1) \to \mathcal{K}_a^\vee \to \mathbf{1},$$

which doesn't split, and so this doesn't have algebraic cycles in characteristic 0. However, in characteristic p this does split, because (fact) $(\mathcal{K}_a)_p = \mathbf{1} \oplus \mathbf{1}(-1)$.

Goal: take the cycle in characteristic p and find its coordinates with respect to de Rham cohomology, i.e., wrt $H^1_{\mathrm{dR}}(\mathbb{G}_{m,\mathbb{Q}}, \{1, a\})$.

For $\mathbb{C}$-periods, the comparison isomorphism

$$H^1_{\mathrm{dR}}(\mathbb{G}_m, \{1, a\}) \xrightarrow{\sim} H^1_B(\mathbb{G}_m, \{1, a\})$$

we get the matrix

	dt	$d\log t$
$1 \to a$	$a-1$	$\log a$
circle	0	$2\pi i$

Let $Z((\mathcal{K}_a)_p) = \mathbb{Q}\cdot v$, where $v \in \mathrm{Hom}(\mathbf{1}, (\mathcal{K}_a)_p^\vee) = \mathrm{Hom}((\mathcal{K}_a)_p, \mathbf{1})$ is the unique map invariant under Frobenius that equals id on $\mathbf{1} \subset (\mathcal{K}_a)_p$. We want to find the coordinates of v in $H^1_{\mathrm{dR}}(\mathbb{G}_m, \{1, a\}) = \mathbb{Q}_p e_0 \oplus \mathbb{Q}_p \; d\log t$. Let $v = (1, \alpha \; d\log t)$; we look for such a vector under Frobenius. We need to compute the matrix of Frobenius acting on $H^1_{\mathrm{dR}}(\mathbb{G}_{m,\mathbb{Q}}, \{1, a\}) \otimes \mathbb{Q}_p$. The relative de Rham cohomology is computed by the complex (the cone over a certain map...)

$$H^\bullet_{\mathrm{dR}}(\mathbb{G}_m, \{1, a\}) = H^\bullet\left(\mathbb{Q}[t^{\pm 1}] \to \mathbb{Q}[t^{\pm 1}]\, dt \oplus \mathbb{Q} \oplus \mathbb{Q}\right),$$

where the non-trivial map in the complex is

$$f \mapsto (df, f(1), -f(a)).$$

Note that $[(df, 0, 0)] = [(0, -f(1), f(a)]$, which is very reminiscent of Stokes's theorem (or the fundamental theorem of calculus).

So now let $e_0 = [(0, 0, 1)]$ and $e_1 = (d\log t, 0, 0)$. Doing the same with $H^\bullet_{\mathrm{rig}}$, we find that the comparison gives

$$H^1_{\mathrm{rig}}(\mathbb{G}_{m,\mathbb{F}_p}, \{1, a\}) = \mathbb{Q}_p\tilde{e}_0 \oplus \mathbb{Q}_p\, d\log t,$$

with the 'obvious' comparison isomorphism ($\tilde{e}_0 \mapsto e_0, d\log t \mapsto d\log t$).

We now lift Frob to $A^\dagger := \mathbb{Z}_p\{t, t^{-1}\} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ by

$$\widetilde{\mathrm{Fr}} = t^p \left(1 + \frac{a^{1-p} - 1}{a - 1}(t-1)\right),$$

where the lift is chosen so that $\widetilde{\mathrm{Frob}}(a) = a$.

We then get $\widetilde{\mathrm{Frob}}^* e_0 = e_0$ and

$$\begin{aligned}\left(\widetilde{\mathrm{Frob}}\right)^*(d\log t) &= d\log\left(t^p\left(1 + \frac{a^{1-p}-1}{a-1}(t-1)\right)\right)\\ &= p\,d\log t + d\left(\sum_{n\geq 0}(-1)^n\frac{1}{n+1}\left(\frac{a^{1-p}-1}{a-1}(t-1)\right)^{n+1}\right).\end{aligned}$$

But in cohomology, $d(g) = [0, -g(1), g(a)]$. When we evaluate at 1 we get 0, so ultimately we get something like $p\,d\log t + \log(a^{1-p})e_0$. So $v = \begin{pmatrix} 1 \\ \frac{\log(\alpha^{1-p})}{1-p}\end{pmatrix}$; we get in particular that $\mathcal{P}_p^{\mathrm{André}}(\mathcal{K}_a^\vee)$.

Theorem 3.4.2.1 (Mahler). *$\log(a) \in \mathbb{Q}_p$ is transcendental, which implies the p-wGPC for $\mathcal{K}_a^\vee$.*

Question 3.4.2.2. *How do we bound* $\operatorname{trdeg}\mathcal{P}_p^{André}(X)$*?*

Let

$$\begin{tikzcd} \mathrm{Mot}(\mathbb{Q})^{\mathrm{gr}} \arrow[r, "\text{fully faithful}"] \arrow[d, "\bmod p"'] & \mathrm{Mot}(\mathbb{Q}) \\ \mathrm{Mot}(\mathbb{F}_p) & \end{tikzcd}$$

Fix $M \in \mathrm{Mot}(\mathbb{Q})$ of good reduction such that $\langle M\rangle$ is abelian. We have a diagram

$$\begin{tikzcd} \langle M\rangle \arrow[rr, "R_{\mathrm{dR}}"] \arrow[d, "\bmod p"'] & & \mathrm{Vec}_{\mathbb{Q}} \arrow[d, "\otimes\mathbb{Q}_p"] \\ \langle M_p\rangle \arrow[rr, "R_{cris}"] \arrow[dr, "Z(-)"'] & & \mathrm{Vec}_{\mathbb{Q}_p} \arrow[dl] \\ & \mathrm{Vec}_{\mathbb{Q}} & \end{tikzcd}$$

Note that Z is not fully faithful, but it is lax-monoidal:

$$Z(A) \otimes Z(B) \to Z(A \times B)$$

is not necessarily an isomorphism. Consider now

$$\begin{tikzcd} \langle M\rangle \arrow[r, "R_{\mathrm{dR}}"] \arrow[dr, "Z_p(-)=Z\circ s_p(-)"'] & \mathrm{Vec}_{\mathbb{Q}} \arrow[r] & \mathrm{Vec}_{\mathbb{Q}_p} \arrow[dl] \\ & \mathrm{Vec}_{\mathbb{Q}} & \end{tikzcd}$$

Theorem 3.4.2.3 (Ancora-Fratila '23). *The functor of points*

$$A \mapsto \mathrm{Emb}^{lax-\otimes}(Z_p|_{\langle M\rangle}, R_{\mathrm{dR}}|_{\langle M\rangle})$$

is representable by an affine variety $\mathcal{H}(M)$, which is homogeneous under $G_{\mathrm{dR}}(M)$. The Berthelot comparison gives a point $c_{\mathrm{Bert}} \in \mathcal{H}(M)(\mathbb{Q}_p)$, which has stabiliser $G_{cris}(M)$. In particular,

$$\dim \mathcal{H} = \dim G_{\mathrm{dR}}(M) - \dim G_{cris}(M_p).$$

Corollary 3.4.2.4. $\operatorname{trdeg} \mathcal{P}_p(\langle M\rangle) \leq \dim \mathcal{H}(M)$.

Chapter 4

Periods (complex and p-adic) and applications IV

4.1 More on André's p-adic periods

Let $X/\overline{\mathbb{Q}}$ be a smooth projective variety with good reduction at p. Let $Z(X_p)$ be the group of algebraic cycles in characteristic p. We have considered the map

$$(\star)\ Z(X_p) \xhookrightarrow[\mathrm{comp}\circ \mathrm{cl}_{cris}]{} H_{\mathrm{dR}}(X/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_p$$

Let $\mathcal{P}_p(X)$ be the $\overline{\mathbb{Q}}$ subalgebra of $\overline{\mathbb{Q}}_p$ generated by the coefficients of this map with respect to $\overline{\mathbb{Q}}$-bases. We also let $\mathcal{P}_p(\langle X\rangle)$ be the subalgebra generated by $\mathcal{P}_p(X^n)$ for all $n \geq 1$. We look for a Tannakian interpretation of these periods.

$$\begin{tikzcd}
\langle X\rangle \arrow[r, "R_{\mathrm{dR}}"] \arrow[d, "\mathrm{sp}"'] & \mathrm{Vec}_{\overline{\mathbb{Q}}} \\
\langle X_p\rangle \arrow[r, "R_{cris}"'] \arrow[dr, "Z(-)"'] & \mathrm{Vec}_{\overline{\mathbb{Q}}_p} \\
& \mathrm{Vec}_{\overline{\mathbb{Q}}}
\end{tikzcd}$$

The first row gives G_{dR} over $\overline{\mathbb{Q}}$; the second, the group $G_{cris}(X_p)$ over $\overline{\mathbb{Q}}_p$.

We also have

$$\begin{tikzcd}
\langle X\rangle \arrow[r, "R_{\mathrm{dR}}"] \arrow[d, "Z_p:=Z\circ\mathrm{sp}"] & \mathrm{Vec}_{\overline{\mathbb{Q}}} \arrow[d] \\
\mathrm{Vec}_{\overline{\mathbb{Q}}} \arrow[r] \arrow[ur] & \mathrm{Vec}_{\overline{\mathbb{Q}}_p},
\end{tikzcd}$$

with the diagonal map $Z_p(-) \otimes \overline{\mathbb{Q}}_p \to R_{\mathrm{dR}}(-) \otimes \overline{\mathbb{Q}}_p$ is $(\star)$.

Note moreover that $(\star)$ is lax-monoidal, that is, there is a commutative diagram

$$\begin{tikzcd}
Z_p(A) \otimes Z_p(B) \arrow[r] \arrow[d, hook] & R_{\mathrm{dR}}(A) \otimes R_{\mathrm{dR}}(B) \arrow[d, "\sim"'] \\
Z_p(A \otimes B) \arrow[r] & R_{dR}(A \otimes B)
\end{tikzcd}$$

where the first vertical arrow is not necessarily an isomorphism (because algebraic cycles don't satisfy Künneth).

Consider all such natural transformations:

$$A \mapsto \mathrm{Emb}^{\otimes}\left(Z_p|_{\langle X\rangle} \otimes_{\overline{\mathbb{Q}}} A, R_{\mathrm{dR}}|_{\langle X\rangle} \otimes_{\overline{\mathbb{Q}}} A\right).$$

Theorem 4.1.0.1 (Ancona-Fratila)**.** *This functor is representable by an affine variety $\mathcal{H}(X)$ over $\overline{\mathbb{Q}}$, which is a homogeneous space under $G_{\mathrm{dR}}(X)$ with stabiliser isomorphic to $G_{cris}(X_p)$.*

Remark 4.1.0.2. To interpret $G_{cris}(X_p)$ inside $G_{\mathrm{dR}}(X)$, use $(\star)$ as point of $\mathcal{H}(X)(\overline{\mathbb{Q}}_p)$.

Corollary 4.1.0.3. $\operatorname{trdeg} \mathcal{P}_p(\langle X\rangle) \leq \dim \mathcal{H}(X) = \dim G_{dR}(X) - \dim G_{cris}(X_p)$.

Conjecture 4.1.0.4 (p-GPC)**.** *This inequality is an equality.*

Example 4.1.0.5.

0. Let $X = \mathbb{G}_m$. We have $G_{\mathrm{dR}}(X) = \mathbb{G}_m$, $G_{cris}(X_p) = \mathbb{G}_m$ (from yesterday), so $\mathcal{H}(X)$ is a point. Moreover, $\mathcal{P}_p(\langle X\rangle) = \overline{\mathbb{Q}}$, which has indeed transcendence degree 0.

1. $X = \mathcal{K}_a^\vee$. We have $R_{\mathrm{dR}}(\mathcal{K}_a^\vee) = H^1_{\mathrm{dR}}(\mathbb{G}_m, \{1, a\})^\vee$, $G_{\mathrm{dR}}(\mathcal{K}_a^\vee) = \mathbb{G}_a \rtimes \mathbb{G}_m$, $G_{cris}((\mathcal{K}_a^\vee)_p) = \mathbb{G}_m$. Hence $\mathcal{H}(\mathcal{K}_a^\vee)$ is of dimension 1. We computed yesterday that $\mathcal{P}_p(\langle \mathcal{K}_a^\vee\rangle) = \overline{\mathbb{Q}}(\log a)$ (for a suitable branch of the p-adic logarithm). By Mahler's theorem, $\log a$ is transcendental, so the p-GPC holds.

2. Let E be an elliptic curve with endomorphism ring M. We work with the motive M. Supersingular means that $M_p = \mathbf{1}^{\oplus 4}$.

	has CM	no CM
ordinary	$G_{\mathrm{dR}} = \mathbb{G}_m, G_{cris} = \mathbb{G}_m$	$G_{\mathrm{dR}} = \mathrm{PGL}_2, G_{cris} = \mathbb{G}_m$
supersingular	$G_{\mathrm{dR}} = \mathbb{G}_m, G_{cris} = 1$	$G_{\mathrm{dR}} = \mathrm{PGL}_2, G_{cris} = 1$

In the four cases (left to right, top to bottom) gives predicted transcendence degrees for $\mathcal{P}_p(\langle M\rangle)$ as follows: $0, 2, 1, 3$.

4.2 Ramified p-adic periods

4.2.1 Application

Theorem 4.2.1.1 (Ancona, Fratila, Vezzani)**.** *Let $p \equiv 1 \pmod 4, g \equiv 1 \pmod 4$, $p \nmid g+1$. There exist infinitely many $a \in \mathbb{Q}(\sqrt{p})$ such that the Jacobian of*

$$C_a : y^2 = x^{2g+2} - a$$

does not descend to $\mathbb{Q}$ up to isogeny, despite the fact that C_a is isomorphic to its Galois conjugate.

Definition 4.2.1.2. Let $L \supset K$ be a field extension, X a variety over L. We say that X *descends* to K if there exists a variety Y/K such that $X \cong Y \times_K L$. An abelian variety A/L *descends up to isogeny to* K if there exists B/K such that $B \times_K L$ is isogenous to A.

Remark 4.2.1.3. Shimura ('72) constructed $C/\mathbb{C}$ such that:

1. $C \cong \overline{C}$ (the complex conjugate);

2. $\mathrm{Jac}(C)$ does not descend to $\mathbb{R}$.

In general, in order to descend a variety, we need at least the variety to be isomorphic to all of its conjugates; but in general, this is not enough, as Shimura's example shows. Huggins ('07) also gave example of curves over number fields which do not descend. Further examples were given by Mestre ('91) and Fité-Florit-Guitart ('24). The latter examples are hyperelliptic curves of genus $g = 2$ over $\mathbb{Q}(\sqrt{2})$ whose Jacobian does not descend to $\mathbb{Q}$ up to isogeny.

Remark 4.2.1.4. Showing that something is not possible / doesn't exist is often very hard! What we need are invariants that can give obstructions.

4.2.2 Ramified periods

Setup 4.2.2.1. Let $L/\mathbb{Q}$ be a number field, totally ramified at p (for example, $L = \mathbb{Q}(\sqrt{p})$). Consider the p-adic completion $\hat{L}/\mathbb{Q}_p$.

Definition 4.2.2.2 (Ramified periods)**.** Let X/L be smooth projective, with good reduction at p. We have the comparison isomorphism

$$H^n_{\mathrm{dR}}(X/L) \otimes_L \hat{L} \to H^n_{cris}(X_p/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \hat{L}.$$

By choosing bases, this isomorphism gives a point in $\mathrm{GL}_N(\hat{L})$. Quotienting out by the choices of bases, we get a double coset in

$$\mathrm{GL}_N(\mathbb{Q}_p) \backslash \mathrm{GL}_N(\hat{L}) / \mathrm{GL}_N(L)$$

denoted by $\mathcal{P}_{\mathrm{ram}}(X, n)$.

Remark 4.2.2.3. The fields L and $\mathbb{Q}_p$ generate $\hat{L}$. Nonetheless, even for $N = 1$, the quotient

$$\mathbb{Q}_p^\times \backslash \hat{L}^\times / L^\times \cong \mathbb{P}_{\mathbb{Q}_p}(\hat{L}) / L^\times$$

is highly nontrivial ($\hat{L}$ is uncountable, $L^\times$ is countable).

Remark 4.2.2.4. If X descends to $\mathbb{Q}$ **with good reduction**, then the ramified periods are trivial. To see this, simply notice that if X descends, we can find an isomorphism

$$H^n_{\mathrm{dR}}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to H^n_{cris}(X_p/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p,$$

whose representative matrix is clearly trivial in the double coset space.

Definition 4.2.2.5. Let V_0 be a $\mathbb{Q}_p$-vector space with a filtration $\mathrm{Fil}^\bullet$ on $V_0 \otimes \hat{L}$. We say that the filtration *descends* to V_0 if

$$\dim_{\mathbb{Q}_p}\left(\mathrm{Fil}^i \cap V_0\right) = \dim_{\hat{L}} \mathrm{Fil}^i .$$

Hypothesis ($\mathrm{Fil_n}$)**.** The Hodge filtration on $H^n_{\mathrm{dR}}(X/L) \otimes \hat{L}$ descends to $H^n_{\mathrm{cris}}(X_p/\mathbb{Q}_p)$.

Remark 4.2.2.6. Note that this is uninteresting in the case $n = 0$ (or top dimension).

Definition 4.2.2.7 (Definition/Proposition)**.** Suppose that X satisfies ($\mathrm{Fil_n}$) for some n. Let $P_{H_0} \leq \mathrm{GL}(H^n_{\mathrm{dR}}(X/L))$ be the parabolic subgroup preserving the Hodge filtration. Simlarly define $P_{cris} \leq \mathrm{GL}(H^n_{cris}(X_p/\mathbb{Q}_p))$. The Berthelot comparison isomorphism gives an element in

$$P_{cris}(\mathbb{Q}_p) \backslash P_{H_0}(\hat{L}) / P_{H_0}(L),$$

denoted by $\mathcal{P}^{\mathrm{fil}}_{\mathrm{ram}}(X, n)$.

Essentially, $\mathcal{P}^{\mathrm{fil}}_{\mathrm{ram}}(X, n)$ is a (block) upper-triangular matrix.

Definition 4.2.2.8. Let $\mu : P_{H_0} \to \mathbb{G}_m$ be a character. Define $\mathcal{P}^\mu_{\mathrm{ram}} := \mu(\mathcal{P}^{\mathrm{fil}}_{\mathrm{ram}}(X, n)) \in \mathbb{Q}_p^\times \backslash \hat{L}^\times / L^\times$.

Example 4.2.2.9. For $P_{H_0} = \left\{ \begin{pmatrix} A & \star \\ 0 & B \end{pmatrix} \right\}$, for all $a, b \in \mathbb{Z}$ the expression $\det(A)^a \cdot \det(B)^b$ gives a possible character μ.

Theorem 4.2.2.10 (Ancona-Fratila-Vezzani '25)**.** *Let X/L be a smooth projective variety with good reduction at p, where $L/\mathbb{Q}$ is a Galois extension, totally ramified at p. Suppose that* (Fil_n) *holds for some n. If X descends to $\mathbb{Q}$, then $P^\mu_{\mathrm{ram}}(X, n)$ is trivial for all characters $\mu : P_{H_0} \to \mathbb{G}_m$.*

Corollary 4.2.2.11. *If there exists μ such that $\mathcal{P}^\mu_{\mathrm{ram}}$ is nontrivial, then X does not descend.*

Proof. Let $Y/\mathbb{Q}$ be a smooth projective variety such that $Y_L \cong X$. The existence of Y produces an action of $\mathrm{Gal}(L/\mathbb{Q})$ on $H_{cris}(X_p/\mathbb{Q}_p)$, and

$$H^n_{\mathrm{dR}}(X/L) \otimes_L \hat{L} \xrightarrow{\sim} H^n_{cris}(X_p/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \hat{L}$$

is $\mathrm{Gal}(L/\mathbb{Q})$-equivariant and compatible with filtrations. Starting from the character, we build a tensor construction in which the filtration gives us lines and a Galois-equivariant isomorphism $\ell_{\mathrm{dR}} \otimes_L \hat{L} \xrightarrow{\sim} \ell_{cris} \otimes_{\mathbb{Q}_p} \hat{L}$.

Example 4.2.2.12.

$$\begin{tikzcd} F^1_{\mathrm{dR}} \subset \arrow[d, "\cong"'] & H^n \arrow[d, "\cong", "\text{Berthelot}"'] \\ F^1_{cris} \subset & H^n; \end{tikzcd}$$

The de Rham and crystalline lines are $\Lambda^{\mathrm{top}}(F^1_{\mathrm{dR}}) \otimes_L \hat{L} \cong \Lambda^{\mathrm{top}}(F^1_{cris}) \otimes_{\mathbb{Q}_p} \hat{L}$. For $P_{H_0} = \left\{ \begin{pmatrix} A & \star \\ 0 & B \end{pmatrix} \right\}$, this corresponds to the character $\det A$.

Now notice that $\hat{L}$ is the regular representation of $\mathrm{Gal}(L/\mathbb{Q})$ and $\ell_{cris} \cong \chi$ is a character of $\mathrm{Gal}(L/\mathbb{Q})$. In particular,

$$\ell_{cris} = (\ell_{cris} \otimes_{\mathbb{Q}_p} \hat{L})^{\mathrm{Gal}(L/\mathbb{Q}),\chi^{-1}}.$$

On the de Rham side, we have

$$\ell_{\mathrm{dR}} = T \otimes_{\mathbb{Q}} L,$$

because $X = Y \times_{\mathbb{Q}} L$, and Galois acts only on L. Since this is essentially the regular representation again, $\ell_{\mathrm{dR}}^{\mathrm{Gal}(L/\mathbb{Q}),\chi^{-1}}$ is nontrivial. Hence $\ell_{\mathrm{dR}} \cap \ell_{\mathrm{cris}}$ is nonzero, and hence, the period is trivial, because (since the $\mathbb{Q}_p$-line and the L-line actually intersect) we can choose bases to make the period literally equal to 1. □

Proposition 4.2.2.13. *Let X/L be smooth projective with good reduction at p. If there exists $Y/\mathbb{Q}_p$ with good reduction such that $X \times_{\mathbb{Q}} \hat{L} \cong Y \times_{\mathbb{Q}_p} \hat{L}$, then* (Fil_n) *is satisfied for all n, and the ramified periods can be computed from a de Rham comparison isomorphism:*

$$\begin{tikzcd} H_{\mathrm{dR}}(X/L) \otimes \hat{L} \arrow[r, "\sim"] \arrow[d, "\cong"'] & H_{\mathrm{dR}}(Y/\mathbb{Q}_p) \otimes \hat{L} \\ H_{cris}(X_p/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \hat{L} \arrow[r, "\sim"'] & H_{cris}(Y_p/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \hat{L} \arrow[u, "\text{Berthelot} \otimes \mathrm{id}"'] \end{tikzcd}$$

The right and bottom isomorphisms are over $\mathbb{Q}_p$. The left and top isomorphisms are over $\hat{L}$. It follows that the double coset classes of the left and top isomorphisms agree. In particular, we can compute the double coset using only the top isomorphism, which is purely on the de Rham side.

4.2.3 Application, in practice

We construct

$$C_a : y^2 = x^{2g+2} - a$$

over $L := \mathbb{Q}_p(\sqrt{p})$ such that $C_a \cong C_{\overline{a}}$, but C_a does not descend, because $\mathcal{P}^{\mu}_{\mathrm{ram}}(C_a, 1) \neq$ trivial. Since

$$\mathcal{P}^{\mu}_{\mathrm{ram}}(\mathrm{Jac}(C_a), 1) = \mathcal{P}^{\mu}_{\mathrm{ram}}(C_a, 1),$$

we obtain that the motive $\mathrm{Jac}(C_a)$ does not descend to $\mathbb{Q}$; by definition of the motive, this means that the variety doesn't descend even up to isogeny.

Construction. Let $p \equiv 1 \pmod 4, g \equiv 1 \pmod 4, p \nmid g+1$. Let $v \in \mathbb{Q}(\sqrt{p})$ satisfy $N(v) = -1$ (Pell equation). Let $b \in \mathbb{Q}(\sqrt{p})$ be such that$v^{g+1} = b/\overline{b}$, which exists by Hilbert 90. Construct $a \in \mathbb{Q}(\sqrt{p})$ as $a = b^2 p^r$, where $r \in \mathbb{Z}$ is chosen so that $v(a) = 0$. Then

$$C_a \cong C_{\overline{a}}.$$

Let $\alpha = \sqrt[2g+2]{a/c} \in \mathbb{Q}_p(\sqrt{p})$, where $c \in \mathbb{Z}$ such that $a \equiv c \pmod{\sqrt{p}}$. The root exists since $p \nmid 2(g+1)$. The ramified period is $\alpha^{-g(g+1)/2}$, which is non-trivial (by direct computation).

Proof. 1. $C_a \cong C_{\overline{a}}$ via the map $(x, y) \mapsto (vx, v^{g+1}y)$.

2. $C_a \times_{\mathbb{Q}_p} \mathbb{Q}_p(\sqrt{p})$ is isomorphic to C_c, and C_c (defined over $\mathbb{Q}_p$, or even over $\mathbb{Q}$) has good reduction since $p \nmid c, p \nmid 2g+2$. The isomorphism is $\phi : (x, y) \mapsto (\alpha x, \alpha^{g+1} y)$.

3. Computation of the period on $\det(\mathrm{Fil}^1)$. By Proposition 4.2.2.13, it suffices to compute

$$\phi^* : H^1_{\mathrm{dR}}(C_a/L) \otimes_L \widehat{L} \cong H^1_{\mathrm{dR}}(C_c/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \hat{L}.$$

   The subspace Fil^1 is spanned by $x^i \frac{dx}{y}$ for $i = 0, \dots, g-1$. By direct computation, $\phi^*(x^i \frac{dx}{y}) = \alpha^{-g+i} x^i \frac{dx}{y}$. Thus, $\det(\phi^*|_{\mathrm{Fil}^1}) = \alpha^{-g(g+1)/2}$.

4. $\alpha^d \in \mathbb{Q}_p^\times \backslash \hat{L}^\times / L^\times$ is non-trivial for every odd d. This uses that $\mathbb{Q}(\sqrt{p})$ is contained in $\mathbb{R}$ and that $N(v) = -1$.

$\square$

Chapter 5

Brauer-Manin obstruction via refined Swan conductors I

5.1 Vague overview & motivation

In this course, we will see how to build classes in the (étale) cohomology of varieties in characteristic 0 (with 'interesting' arithmetic properties) starting from (de Rham) cohomology classes in positive characteristic.

Let V/M be a variety over a number field. We are interested in the set $V(M)$ of M-rational points of V.

Remark 5.1.0.1. In this course, *variety over* M means *smooth, proper, geometrically integral* M*-scheme*. Typically, V will be a projective variety $V = V(f_1, \ldots, f_m) \subset \mathbb{P}^n_M$, and

$$V(M) = \{(a_0 : \cdots : a_n) \in \mathbb{P}^n(M) \mid f_i(a_0, \ldots, a_n) = 0 \ \forall i = 1, \ldots, r\}.$$

In general, $V(M)$ is extremely hard to understand.

To study $V(M)$, it often helps to look at the inclusion

$$V(M) \hookrightarrow \prod_{v \in \Omega_M} V(M_v),$$

where Ω_M is the set of places of M and M_v is the completion at v.

Example 5.1.0.2. For $M = \mathbb{Q}$, we have $\Omega_{\mathbb{Q}} = \{\text{primes}\} \cup \{\infty\}$ and, for $v \in \Omega_{\mathbb{Q}}$, the completion $\mathbb{Q}_v$ is

$$\mathbb{Q}_v = \begin{cases} \mathbb{Q}_p, & \text{if } v \text{ corresponds to the prime } p \\ \mathbb{R} = \mathbb{Q}_\infty, & \text{if } v \text{ is the place at } \infty. \end{cases}$$

However, $\prod_{v \in \Omega_M} V(M_v)$ is often too big to give useful information on $V(M)$.

Idea (Manin, 1971): use the Brauer group of V, namely $\mathrm{Br}(V) := H^2_{\text{ét}}(V, \mathbb{G}_m)$, to build, for every $\mathcal{A} \in \mathrm{Br}(V)$, an embedding

$$V(M) \hookrightarrow \left(\prod_{v \in \Omega_M} V(M_v) \right)^{\mathcal{A}} \subset \prod_{v \in \Omega_M} V(M_v),$$

where one might hope that the second inclusion is strict.

Our objective is to build 'interesting' Brauer classes, namely, elements $\mathcal{A} \in \mathrm{Br}(V)$ such that

$$V(M) \hookrightarrow \left(\prod_{v \in \Omega_M} V(M_v) \right)^{\mathcal{A}} \subsetneq \prod_{v \in \Omega_M} V(M_v).$$

We will see how, for a prime ideal $\mathfrak{p} \lhd \mathcal{O}_M$ of good reduction for V, we can construct a map

$$\left\{ \begin{array}{c} \text{global 2-forms on the} \\ \text{reduction of } V \text{ modulo } \mathfrak{p} \end{array} \right\} \longrightarrow \{\text{interesting Brauer classes}\}.$$

Definition 5.1.0.3. We say that V has good reduction at $\mathfrak{p}$ is there exists a smooth and proper scheme $\mathcal{X}/\mathcal{O}_K$ such that $\mathcal{X} \times_{\mathcal{O}_{M_\mathfrak{p}}} M_\mathfrak{p} \cong V \times_M M_\mathfrak{p}$.

Example 5.1.0.4. In practice, this is easier than it sounds. Let $V = V(f_1, \ldots, f_n) \subset \mathbb{P}^n_\mathbb{Q}$, with $f_i \in \mathbb{Q}[x_0, \ldots, x_n]$. Assume that $p \in \mathbb{Z}$ is a prime number such that each f_i is in $\mathbb{Z}_p[x_0, \ldots, x_n]$ (that is, p does not appear in the denominator of the defining equations; this can always be arranged, for example by rescaling the f_i). Let $\mathcal{X} = Z(f_1, \ldots, f_n) \subset \mathbb{P}^n_{\mathbb{Z}_p}$. If $\mathcal{X}$ is smooth, then p is a prime of good reduction, using $\mathcal{X}$ as integral model.

Remark 5.1.0.5. Let X/K be smooth and proper, where K is a p-adic field (a finite extension of $\mathbb{Q}_p$), and let $\mathcal{X}$ be an integral model of good reduction (assumed to exist). Let $k = \mathcal{O}_K/\mathfrak{p}$ be the residue field and $Y := \mathcal{X} \times_{\mathcal{O}_K} k$ be the special fibre. In 2016, Bhatt-Morrow-Scholze showed

$$\dim_{\overline{k}} H^2_{\mathrm{dR}}\left(Y_{\overline{k}}/\overline{k}\right) \geq \dim_{\mathbb{F}_p} H^2_{\text{ét}}(X_{\mathbb{C}_p}, \mu_p).$$

We will see how elements in $H^0(Y, \Omega^2_Y)$ “ $\subseteq$ ” $H^2_{\mathrm{dR}}(Y_{\overline{k}}/\overline{k})$ give rise to classes in $H^2_{\text{ét}}(X_{\overline{K}}, \mu_p)$, which in turn will give elements of $\mathrm{Br}(X)[p]$.

5.2 The Brauer-Manin set

5.2.1 First definitions

Definition 5.2.1.1. Given an M-scheme S, the Brauer group of S is

$$\mathrm{Br}(S) := H^2_{\text{ét}}(S, \mathbb{G}_m).$$

We consider Br as a functor $(\mathbf{Sch}_M)^{\mathrm{op}} \to \mathbf{Ab}$.

Given an M-algebra R and an element $\mathcal{A} \in \mathrm{Br}(V)$, functoriality of Br gives an evaluation map

$$\begin{array}{cccc} \mathrm{ev}_\mathcal{A} : & V(R) & \to & \mathrm{Br}(R) := \mathrm{Br}(\mathrm{Spec}(R)) \\ & (P : \mathrm{Spec}(R) \to V) & \mapsto & P^*(\mathcal{A}), \end{array}$$

where $P^* := \mathrm{Br}(P) : \mathrm{Br}(V) \to \mathrm{Br}(R)$ is the map induced by functoriality.

There is a commutative diagram

$$\begin{tikzcd} V(M) \arrow[r] \arrow[d, "\mathrm{ev}_\mathcal{A}"'] & \prod_{v\in\Omega_M} V(M_v) \arrow[d] \arrow[dr, "(\mathrm{ev}_v)_{v\in\Omega_M}"] & \\ \mathrm{Br}(M) \arrow[r] & \bigoplus_{v\in\Omega_M} \mathrm{Br}(M_v) \arrow[r] & \prod_{v\in\Omega_M} \mathrm{Br}(M_v). \end{tikzcd}$$

We then set

$$\left(\prod_{v\in\Omega_M} V(M_v)\right)^{\mathcal{A}} := \left\{ (x_v) \in \prod_{v\in\Omega_M} V(M_v) \mid (\mathrm{ev}_\mathcal{A}(x_v)) \in \mathrm{Im}\left(\mathrm{Br}(M) \to \bigoplus_{v\in\Omega_M} \mathrm{Br}(M_v)\right)\right\}.$$

5.2.2 Results from class field theory

We discuss the Brauer group of local and global fields of characteristic 0.

Theorem 5.2.2.1 (Class field theory)**.** *The following hold.*

1. *For local fields we have*

$$\mathrm{Br}(M_v) \xrightarrow[\mathrm{inv_v}]{\sim} \begin{cases} \mathbb{Q}/\mathbb{Z}, & \text{if } M_v \text{ is } p\text{-adic} \\ \frac{1}{2}\mathbb{Z}/\mathbb{Z}, & \text{if } M_v \cong \mathbb{R} \\ 0, & \text{if } M_v \cong \mathbb{C}. \end{cases}$$

2. *(Albert-Brauer-Hasse-Noether) Let M be a number fields. There is a short exact sequence*

$$0 \to \mathrm{Br}(M) \to \bigoplus_{v\in\Omega_M} \mathrm{Br}(M_v) \to \sum \mathbb{Q}/\mathbb{Z} \to 0.$$

Using Albert-Brauer-Hasse-Noether, we find

$$\left(\prod_{v\in\Omega_M} V(M_v)\right)^{\mathcal{A}} = \left\{(x_v)\in\prod_{v\in\Omega_M} \mid \sum_v \mathrm{inv}_v(\mathrm{ev}_{\mathcal{A}}(x_v)) = 0\right\}.$$

Exercise 5.2.2.2. *Assume $\prod_{v\in M_v} V(M_v)\neq\emptyset$ (this is relatively easy to check in practice). If $\mathcal{A}\in\mathrm{Br}(V)$ is such that there exists $v\in\Omega_v$ for which $\mathrm{ev}_{\mathcal{A}}: V(M_v)\to\mathrm{Br}(M_v)$ is non-constant, then*

$$\left(\prod_{v\in M_v} V(M_v)\right)^{\mathcal{A}} \subsetneq \prod_{v\in M_v} V(M_v),$$

so $\mathcal{A}$ is an 'interesting' Brauer class.

Remark 5.2.2.3. It may happen that $\mathrm{ev}_{\mathcal{A}}$ is constant on $V(M_v)$ for all v: if there is a rational point, then (by evaluating at this point) we find $\left(\prod_{v\in M_v} V(M_v)\right)^{\mathcal{A}} = \prod_{v\in M_v} V(M_v)$. Otherwise, it is possible that $\left(\prod_{v\in M_v} V(M_v)\right)^{\mathcal{A}} = \emptyset$. When $\mathrm{ev}_{\mathcal{A}}$ is non-constant, $\mathcal{A}$ cannot fully obstruct the existence of rational points.

Remark 5.2.2.4. If there exists $\mathcal{A}\in\mathrm{Br}(V)$ such that there exists $\mathfrak{p}\lhd\mathcal{O}_M$ for which $\mathrm{ev}_{\mathcal{A}}: V(M_p)\to\mathrm{Br}(M_p)$ is non-constant, then there exists $\mathcal{A}'\in\mathrm{Br}(V_{M_\mathfrak{p}})$ such that $\mathrm{ev}_{\mathcal{A}'}: V_{M_\mathfrak{p}}(M_\mathfrak{p})\to\mathrm{Br}(M_\mathfrak{p})$ is non-constant (exercise). The converse is also true, up to replacing M with a finite extension M'. This motivates the setting for the rest of this course.

5.3 Setup

We let $K/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_K$ and residue field k. We will denote by $\mathcal{X}$ a smooth and proper $\mathcal{O}_K$-scheme with generic fibre $X = \mathcal{X}\otimes_{\mathcal{O}_K} K$, which is smooth and proper over K, and special fibre $Y = \mathcal{X}\times_{\mathcal{O}_K} k$, which is smooth and proper over k.

We want to understand whether, given $\mathcal{A}\in\mathrm{Br}(X)$, the evaluation map $\mathrm{ev}_{\mathcal{A}}: X(K)\to\mathrm{Br}(K)$ is non-constant.

Remark 5.3.0.1. If $\mathcal{A}\in\mathrm{Br}(X)$ comes from $\mathrm{Br}(\mathcal{X})$, the evaluation map $\mathrm{ev}_{\mathcal{A}}$ is constant (and trivial). To see this, assume that $\mathcal{A}$ lies in the image of $\mathrm{Br}(\mathcal{X})\to\mathrm{Br}(X)$. Then we have a commutative diagram

$$\begin{tikzcd} X(K) \arrow[r, "\mathrm{ev}_{\mathcal{A}}"] \arrow[d, "="'] & \mathrm{Br}(K) \\ \mathcal{X}(\mathcal{O}_K) \arrow[r, "\mathcal{A}"'] & \mathrm{Br}(\mathcal{O}_K) \arrow[u] \end{tikzcd}$$

Now since $\mathcal{O}_K$ is Henselian we have $\mathrm{Br}(\mathcal{O}_K)=\mathrm{Br}(k)$, and the Brauer group of finite fields is trivial. Thus, $\mathrm{ev}_{\mathcal{A}}$ factors through the trivial group, hence it is (constant and) trivial.

5.4 The prime-to-p case

Theorem 5.4.0.1. *Let n be coprime to p. The following hold.*

1. *$\mathcal{A}\in\mathrm{Br}(X)[n]$ has trivial evaluation map for all finite extensions K'/K if and only if $\mathcal{A}$ comes from the Brauer group of the model, which we write as $\mathcal{A}\in\mathrm{Br}(\mathcal{X})[n]$.*

2. *For every $\mathcal{A}\in\mathrm{Br}(X)[n]$, there exists a finite unramified extension K'/K such that $\mathcal{A}\in\mathrm{Br}(\mathcal{X}_{\mathcal{O}_{K'}})[n]$.*

Sketch of proof. We have a reduction map

$$(-)_0 : X(K) = \mathcal{X}(\mathcal{O}_K)\to\mathcal{X}(k) = Y(k)$$

and a residue map

$$\partial_n : \mathrm{Br}(X)[n]\to H^1_{\text{ét}}(Y,\mathbb{Z}/n\mathbb{Z})$$

such that for every $P \in X(K)$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{Br}(X)[n] \arrow[r, "\partial_n"] \arrow[d, "P^*"'] & H^1_{\text{ét}}(Y, \mathbb{Z}/n\mathbb{Z}) \arrow[d, "P_0^*"] \\ \mathrm{Br}(K)[n] \arrow[r, "\sim"'] & H^1(k, \mathbb{Z}/n\mathbb{Z}). \end{tikzcd}$$

The lower isomorphism comes from local class field theory.

The purity theorem for the Brauer group gives an exact sequence

$$0 \to \mathrm{Br}(\mathcal{X})[n] \to \mathrm{Br}(X)[n] \xrightarrow{\partial_n} H^1(Y, \mathbb{Z}/n\mathbb{Z}).$$

1. Assume that $\mathcal{A} \in Br(X)[n]$ does not come from the model. Then $\partial_n \mathcal{A}$ is some non-trivial element $[Y'] \in H^1_{\text{ét}}(Y_{\overline{k}}, \mathbb{Z}/n\mathbb{Z})$. There exist a finite extension k' of k and a point $Q_0 \in Y(k')$ such that $Q_0^*([Y'])$ is non-trivial (not easy; uses Chebotarev-like arguments). Using the commutative diagram above, this gives the existence of $Q \in X(K')$ such that $Q^*(\mathcal{A})$ is nontrivial, where K'/K is the finite extension corresponding to k'/k.

2. The Kummer sequence on $X_{\text{ét}}$

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{\bullet \mapsto \bullet^n} \mathbb{G}_m \to 1$$

gives the exact sequence

$$0 \to \mathrm{Pic}(X)/n\,\mathrm{Pic}(X) \to H^2_{\text{ét}}(X, \mu_n) \to \mathrm{Br}(X)[n] \to 0;$$

since n is coprime to p, we have the same sequence also for $\mathcal{X}$, and hence a sequence

$$0 \to \mathrm{Pic}(\mathcal{X})/n\,\mathrm{Pic}(\mathcal{X}) \to H^2_{\text{ét}}(\mathcal{X}, \mu_n) \to \mathrm{Br}(\mathcal{X})[n] \to 0;$$

with obvious connecting vertical arrows (from the sequence for $\mathcal{X}$ to the sequence for X). One can show that $\mathrm{Pic}(\mathcal{X})/n\,\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X)/n\,\mathrm{Pic}(X)$ is an isomorphism, and (using smoothness and properness) we also have

$$H^2_{\text{ét}}(X_{\overline{K}}, \mu_n) \cong H^2_{\text{ét}}(\mathcal{X}_{\mathcal{O}_{K'}}, \mu_n).$$

An easy diagram chase now gives the claim.

$\square$

Chapter 6

Brauer-Manin obstruction via refined Swan conductors II

6.1 Recap

Recall our standing notation: K is a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_K$ is its ring of integers, π a uniformiser of K, and $k := \mathcal{O}_K/(\pi)$ the residue field. Let $\mathcal{X}$ be a smooth proper scheme over $\mathcal{O}_K$, and let X, respectively Y, be the generic and special fibres of $\mathcal{X}$, respectively.

Remark 6.1.0.1. Note that $X \subset \mathcal{X}$ is open, with complement Y.

We want to understand under what conditions an element $\mathcal{A} \in \mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$ has a non-constant evaluation map. We showed that, for every integer n coprime to p, every element $\mathcal{A} \in \mathrm{Br}(X)[n]$ has constant evaluation map on $X(K')$, for all finite extensions K'/K, if and only if $\mathcal{A} \in \mathrm{Br}(\mathcal{X})[n]$. The key tool in the proof was the exact sequence

$$0 \to \mathrm{Br}(\mathcal{X})[n] \to \mathrm{Br}(X)[n] \to H^1_{\text{ét}}(Y, \mathbb{Z}/n\mathbb{Z}).$$

6.2 Some explicit computations

Fact 6.2.0.1. *The following facts are useful to describe the Brauer group of a variety.*

- *Let $K(X)$ be the function field of X. There is a canonical embedding* $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(K(X))$*. Typically, one describes elements of* $\mathrm{Br}(X)$ *via their image in* $\mathrm{Br}(K(X))$*.*
- $\mathrm{Br}(K(X))$ *can be identified with the set of central simple algebras* $\{\mathrm{CSA}_{K(X)}\}/\sim$*. A central simple algebra (CSA) over $K(X)$ is a finite associative $K(X)$-algebra with center $K(X)$ and whose only two-sided ideals are the trivial ones. The equivalence relation is $A \sim B$ if there exist $m, n \geq 1$ such that $A \otimes_{K(X)} M_n(K(X)) \cong B \otimes_{K(X)} M_m(K(X))$. The group operation is (induced by) tensor product.*

Example 6.2.0.2 (Quaternion algebras). Let F be a field and $f, g \in F^\times$. We define $\mathcal{Q}(f, g)$ is the 4-dimensional F-algebra with basis $1, i, j, ij$ and multiplicative structure

$$i^2 = f,\ j^2 = g,\ ij = -ji.$$

This is a central simple algebra over F.

Definition 6.2.0.3. We denote by (f, g) the equivalence class of $[\mathcal{Q}(f, g)] \in \mathrm{Br}(F)$.

Fact 6.2.0.4. *Quaternion algebras are of order* 2 *in the Brauer group.*

We make explicit the constructions considered so far in the case of quaternion algebras.

1. Evaluation map. Let $\mathcal{A} \in \mathrm{Br}(X)$ be an element such that $\mathcal{A}_{K(X)} \in \mathrm{Br}(K(X))$ is of the form (f, g) for some $f, g \in K(X)^\times$. If P is a K-point such that f, g are both regular and non-vanishing at P, the evaluation map is simply $\mathrm{ev}_{\mathcal{A}}(P) = (f(P), g(P)) \in \mathrm{Br}(K)$.

Remark 6.2.0.5. If $\mathcal{A} \in \mathrm{Br}(X)$ maps to a quaternion algebra in $\mathrm{Br}(K(X))$, for every $P \in X(K)$ there are $f, g \in K(X)^\times$, regular and non-vanishing at P, such that $\mathcal{A}_{K(X)} = (f, g)$.

2. Residue map $\partial_n = \partial_2$. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Br}(X)[2] \arrow[r, "\partial_2"] \arrow[d, hook] & H^1_{\text{ét}}(Y, \mathbb{Z}/2\mathbb{Z}) \arrow[d, hook] \\ \mathrm{Br}(X(K))[2] \arrow[r] & H^1_{\text{ét}}(k(Y), \mathbb{Z}/2\mathbb{Z}) \end{tikzcd}$$

Recall that $Y \subset \mathcal{X}$ is a closed irreducible subscheme of codimension 1. Thus, $\mathcal{O}_{\mathcal{X},\eta_Y}$ is a regular local ring of dimension 1, hence a DVR, and so we have a valuation $\nu_Y : K(X) \to \mathbb{Z}$. One can then show that the residue map is

$$\begin{array}{rccc} \partial_2 : & \mathrm{Br}(K(X))[2] & \to & H^1(k(Y), \mathbb{Z}/2\mathbb{Z}) \cong k(Y)^\times / k(Y)^{\times 2} \\ & (f, g) & \mapsto & \left[(-1)^{\nu_Y(f)\nu_Y(g)} \frac{f^{\nu_Y(g)}}{g^{\nu_Y(f)}}\right]. \end{array}$$

Example 6.2.0.6 (An affine elliptic curve). Let $E/\mathbb{Q}_p$, with $p \neq 2$, be given by $E : y^2 = x(x-a)(x-b)$, with $a, b \in \mathbb{Z}_p^\times$ and $\overline{a}, \overline{b} \in \mathbb{Z}/p\mathbb{Z}$ distinct (and nonzero). We use the obvious integral model $\mathcal{E}/\mathbb{Z}_p : y^2 = x(x-a)(x-b)$ and have the special fibre $\tilde{E} : y^2 = x(x-\overline{a})(x-\overline{b})$. The function field of E is simply

$$\mathbb{Q}_p(E) = \mathrm{Frac}\left(\frac{\mathbb{Q}_p[x, y]}{(y^2 - x(x-a)(x-b))}\right).$$

We think of $\tilde{E} \subset \mathcal{E} = \mathrm{Spec}\, \frac{\mathbb{Z}_p[x,y]}{(y^2 - x(x-a)(x-b))}$ as the zero locus of the ideal (p). In particular, $\mathcal{O}_{\mathcal{E},\eta_{\tilde{E}}} = \left(\frac{\mathbb{Z}_p[x,y]}{(y^2 - x(x-a)(x-b))}\right)_{(p)}$.

Fact 6.2.0.7. *One can show that, for every $\lambda \in \mathbb{Q}_p$, the class of $(\lambda, x-a) \in \mathrm{Br}(\mathbb{Q}_p(E))$ comes from $\mathrm{Br}(\mathcal{E})[2]$. We will later see how to prove such statements.*

Exercise 6.2.0.8. *Prove that $\nu_{\tilde{E}}(x-a) = 0$ and $\nu_{\tilde{E}}(\lambda) = \nu_p(\lambda)$.*

As a consequence, $\partial_2((\lambda, x-a)) = [(x-a)^{v_p(\lambda)}] \in \frac{\mathbb{F}_p(\tilde{E})^\times}{\mathbb{F}_p(\tilde{E})^{\times 2}}$.

Exercise 6.2.0.9. *The class $[(x-a)]$ is not in $\mathbb{F}_p(\tilde{E})^{\times 2}$.*
Hint. Look at its divisor.

Thus, the residue is trivial if and only if $\nu_p(\lambda)$ is even.

Remark 6.2.0.10.

- $\partial_2(\lambda, x-a) = 0$ if and only if $\nu_p(\lambda)$ is even.
- Over $\mathbb{Q}_p(\sqrt{\lambda})$, we will have $\partial_2(\lambda, x-a) = [(x-a)^{\nu_\pi(\lambda)}] = 0$, so $(\lambda, x-a)$ comes from the model over $\mathcal{X}_{\mathcal{O}_{\mathbb{Q}_p(\sqrt{\lambda})}}$.

6.3 The purity sequence

Let's consider again the sequence

$$0 \to \mathrm{Br}(\mathcal{X})[n] \to \mathrm{Br}(X)[n] \to H^1_{\text{ét}}(Y, \mathbb{Z}/n\mathbb{Z}).$$

This is a form of the purity theorem.

Theorem 6.3.0.1 (Purity)**.** *We have*

$$\mathrm{Br}(X) = \bigcap_{x \text{ of codimension } 1} \mathrm{Br}(\mathcal{O}_{X,x})$$

and

$$\mathrm{Br}(\mathcal{X}) = \bigcap_{x \text{ of codimension } 1} \mathrm{Br}(\mathcal{O}_{\mathcal{X},x}).$$

Remark 6.3.0.2. In particular, the Brauer group doesn't change if we remove points of codimension at least 2.

Corollary 6.3.0.3. *Since the only codimension-1 point of $\mathcal{X}$ that is not also a codimension-1 point of X is the special fibre Y, we obtain*

$$\mathrm{Br}(\mathcal{X}) = \mathrm{Br}(X) \cap \mathrm{Br}(\mathcal{O}_{X,\eta_Y}),$$

where the intersection is taken inside $\mathrm{Br}(K(X))$*, as in the following diagram*

$$\begin{array}{ccc} \mathrm{Br}(\mathcal{X}) & \longrightarrow & \mathrm{Br}(X) \\ \downarrow & & \downarrow \\ \mathrm{Br}(\mathcal{O}_{X,\eta_Y}) & \longrightarrow & \mathrm{Br}(K(X)) \end{array}$$

Construction of the residue map

Let R be the DVR $\mathcal{O}_{\mathcal{X},\eta_Y}$, let R^h be its Henselisation, and denote by K^h the fraction field of R^h.

Remark 6.3.0.4. Recall the following properties of Henselisation: the ring R^h is Henselian; there is a map $R \to R^h$, and R, R^h have the same residue field. More precisely, if $\mathfrak{m}$ is the maximal ideal of R, then $\mathfrak{m}R^h$ is the maximal ideal of R^h, and the natural map $R/\mathfrak{m} \to R^h/\mathfrak{m}R^h$ is an isomorphism. In our case, this residue field is $k(Y)$.

Fact 6.3.0.5. *1. We have* $\mathrm{Br}(R) = \mathrm{Br}(K(X)) \cap \mathrm{Br}(R^h)$.

2. Let n be coprime to p. We have

$$\mathrm{Br}(K^h)[n] = \ker\left(\mathrm{Br}(K^h)[n] \to \mathrm{Br}(K^h_{\mathrm{nr}})[n]\right):$$

every n-torsion element in $\mathrm{Br}(K^h)$ is trivialised by an unramified extension.

3. Using some Galois cohomology,the short exact sequence

$$1 \to U_{\mathrm{nr}} \to (K^h_{\mathrm{nr}})^\times \xrightarrow{\nu} \mathbb{Z} \to 1$$

yields

$$0 \to \mathrm{Br}(R^h) \to \ker\left(\mathrm{Br}(K^h) \to \mathrm{Br}(K^h_{\mathrm{nr}})\right) \xrightarrow{\partial} H^1(k(Y), \mathbb{Q}/\mathbb{Z}) \to 0$$

(This is part of the low-degree exact sequence; we note that

$$\mathrm{Br}(R^h) = H^2(G_{\mathrm{nr}}, U_{\mathrm{nr}}), \qquad \ker\left(\mathrm{Br}(K^h) \to \mathrm{Br}(K^h_{\mathrm{nr}})\right) = H^2(G_{\mathrm{nr}}, (K^h_{\mathrm{ur}})^\times)$$

and $H^1(k(Y), \mathbb{Q}/\mathbb{Z}) \cong H^2(k(Y), \mathbb{Z})$ via the usual short exact sequence coming from $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$.)

Now the point is that for n coprime to p we have $\ker\left(\mathrm{Br}(K^h)[n] \to \mathrm{Br}(K^h_{\mathrm{nr}})[n]\right) = \mathrm{Br}(K^h)[n]$, and so the residue is defined everywhere.

Putting everything together, we have the following.

Lemma 6.3.0.6. *If $(n, p) = 1$, then there exists a residue map*

$$\partial_n : \mathrm{Br}(K(X))[n] \to H^1(k(X), \mathbb{Z}/n\mathbb{Z})$$

such that $\ker(\partial_n) = \mathrm{Br}(\mathcal{O}_{X,\eta_Y})[n]$.

Proof. We define ∂_n as the composition

$$\mathrm{Br}(K(X))[n] \to \mathrm{Br}(K^h)[n] \xrightarrow{\partial} H^1(k(Y), \mathbb{Z}/n\mathbb{Z}),$$

where ∂ exists by facts (2) and (3). We now have $\partial_n(\mathcal{A}) = 0$ if and only if $\partial_n(\mathcal{A}_{K^h}) = 0$, if and only if $\mathcal{A}_{K^h} \in \mathrm{Br}(R^h)$ (by fact (3)), if and only if $\mathcal{A} \in \mathrm{Br}(R)$ (by fact (1)), as desired. $\square$

Definition 6.3.0.7. We define

$$\mathrm{fil}_0 \mathrm{Br}(X) = \{\mathcal{A} \in \mathrm{Br}(X) \mid \mathcal{A}_{K^h} \in \ker\left(\mathrm{Br}(K^h) \to \mathrm{Br}(K^h_{\mathrm{nr}})\right)\}.$$

Note that $\mathrm{fil}_0 \mathrm{Br}(X)[n] = \mathrm{Br}(X)[n]$ for n coprime to p, and that the residue map is defined on $\mathrm{fil}_0 \mathrm{Br}(X)$ (even for p-power torsion in the Brauer group).

Chapter 7

Brauer-Manin obstruction via refined Swan conductors III

Let $K/\mathbb{Q}_p$ be a finite extension, X/K be a smooth and proper variety, $L := K(X)$. We defined a subgroup

$$\mathrm{fil}_0 \operatorname{Br}(X) \subseteq \operatorname{Br}(X)$$

given by

$$\mathrm{fil}_0 \operatorname{Br}(X) = \{\mathcal{A} \in \operatorname{Br}(X) \bigm| \mathcal{A}_{L^h} \in \ker(\operatorname{Br}(L^h) \to \operatorname{Br}(L^h_{\mathrm{nr}})\}.$$

Lemma 7.0.0.1. *There is a residue map*

$$\partial : \mathrm{fil}_0 \operatorname{Br}(X) \to H^1(Y, \mathbb{Q}/\mathbb{Z})$$

such that

$$\begin{array}{ccc} \mathrm{fil}_0 \operatorname{Br}(X) & \longrightarrow & H^1(Y, \mathbb{Q}/\mathbb{Z}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{fil}_0 \operatorname{Br}(L^h) & \longrightarrow & H^1(k(Y), \mathbb{Q}/\mathbb{Z}). \end{array}$$

Moreover, $\ker \partial = \operatorname{Br}(\mathcal{X})$.

Theorem 7.0.0.2. *1. An element $\mathcal{A} \in \operatorname{Br}(X)$ has trivial evaluation map on $X(K')$ for all finite extensions K'/K if and only if $\mathcal{A}$ is in $\operatorname{Br}(\mathcal{X})$.*

2. For every element in $\mathrm{fil}_0 \operatorname{Br}(X)$, there exists a finite extension K'/K such that $\mathcal{A} \in \operatorname{Br}(\mathcal{X}_{\mathcal{O}_{K'}})$

Proof. Part (1) is proved as in the prime-to-p case. For part (2), the argument we gave for the prime-to-p torsion does not work; one needs the explicit description of the residue map. □

7.1 The wild part of the Brauer group

We describe some recent work of Bright and Newton (2020).

7.1.1 Step 1.

Kato defined a filtration $\{\mathrm{fil}_n \operatorname{Br}(L^h)\}_{n\in\mathbb{N}}$ such that

1. $\bigcup_{n\in\mathbb{N}} \mathrm{fil}_n \operatorname{Br}(L^h) = \operatorname{Br}(L^h)$
2. For each $n \geq 1$, there exists a map ('refined Swan conductor')

$$\mathrm{rsw}_{n,\pi} : \mathrm{fil}_n \operatorname{Br}(L^h) \to \Omega^2_{k(Y)} \oplus \Omega^1_{k(Y)}$$

such that $\ker(\mathrm{rsw}_{n,\pi}) = \mathrm{fil}_{n-1} \operatorname{Br}(L^h)$.

Remark 7.1.1.1. The map $\mathrm{rsw}_{n,\pi}$ depends on the choice of uniformiser π, but the resulting filtration does not (different choices of π give maps $\mathrm{rsw}_{n,\pi}$ that only differ by multiplication by a scalar, so their kernels don't change).

7.1.2 Step 2.

Bright and Newton define

$$\mathrm{fil}_n \mathrm{Br}(X) = \{\mathcal{A} \in \mathrm{Br}(X) \mid \mathcal{A}_{L^h} \in \mathrm{fil}_n \mathrm{Br}(L^h)\}$$

and show that there exists a diagram

$$\begin{tikzcd} \mathrm{fil}_n \mathrm{Br}(X) \arrow[r, "\mathrm{rsw}_{n,\pi}"] \arrow[d] & H^0(\Omega^2_Y) \oplus H^0(\Omega^1_Y) \arrow[d] \\ \mathrm{fil}_n \mathrm{Br}(L^h) \arrow[r] & \Omega^2_{k(Y)} \oplus \Omega^1_{k(Y)} \end{tikzcd}$$

such that $\ker \mathrm{rsw}_{n,\pi} = \mathrm{fil}\, n-1\, \mathrm{Br}(X)$.

Theorem 7.1.2.1 (Bright & Newton)**.** *Let $\mathcal{A} \in \mathrm{Br}(X)$ be such that $\mathcal{A} \in \mathrm{fil}_n \mathrm{Br}(X)$ and $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta)$. If there exists a point $P \in Y(k)$ such that $(\alpha_P, \beta_P) \neq (0,0)$, then $\mathrm{ev}_{\mathcal{A}} : X(K) \to \mathrm{Br}(K)$ is non-constant.*

Remark 7.1.2.2. In particular, if $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta) \neq (0,0)$, there exists a finite unramified extension K'/K such that $\mathrm{ev}_{\mathcal{A}} : X(K') \to \mathrm{Br}(K')$ is non-constant.

Goal. Give an explicit description for $\mathrm{fil}_n \mathrm{Br}(X)[p]$.

7.2 Elements in $\mathrm{Br}(L)[p]$ for a field L

Let L be a field of characteristic 0 and $n \in \mathbb{Z}_{\geq 0}$. There is an exact sequence of étale sheaves

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m \to 1,$$

which (taking the long exact sequence in cohomology and using Hilbert 90) gives

$$\begin{cases} H^2(L, \mu_n) \xrightarrow{\sim} \mathrm{Br}(L)[n] \\ L^\times \xrightarrow{\delta} H^1(L, \mu_n). \end{cases}$$

Definition 7.2.0.1. For every $j \geq 1$ we define $\mathbb{Z}/n\mathbb{Z}(j)$ as $\underbrace{\mu_n \otimes_{\mathbb{Z}} \mu_n \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mu_n}_{j \text{ times}}$. We also set $\mathbb{Z}/n\mathbb{Z}(0) := \mathbb{Z}/n\mathbb{Z}$. These are all isomorphic as groups, but not as Galois modules.

Exercise 7.2.0.2. *If L contains an n-th root of unity ζ_n, the map $\theta_{\zeta_n} : \mathbb{Z}/n\mathbb{Z}(0) \to \mathbb{Z}/n\mathbb{Z}(1)$ sending 1 to ζ_n is an isomorphism of Galois modules.*

The natural map $\mathbb{Z}/n\mathbb{Z}(i) \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}(j) \to \mathbb{Z}/n\mathbb{Z}(i+j)$ gives a cup product

$$H^1(L, \mathbb{Z}/n\mathbb{Z}(i)) \times H^1(L, \mathbb{Z}/n\mathbb{Z}(j)) \to H^2(L, \mathbb{Z}/n\mathbb{Z}(i+j)).$$

In particular, if L contains a primitive n-root of unity ζ_n, we get

$$H^1(L, \mathbb{Z}/n\mathbb{Z}(1)) \times H^1(L, \mathbb{Z}/n\mathbb{Z}(1)) \to H^2(L, \mathbb{Z}/n\mathbb{Z}(2)) \cong H^2(L, \mathbb{Z}/n\mathbb{Z}(1)) = \mathrm{Br}(L)[n]. \tag{7.2.0.3}$$

7.3 The ramification filtration on $\mathrm{Br}(L^h)[p]$

Assume that L^h contains a p-th root of unity. Bloch & Kato define a filtration $U^m H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2))$ on $H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2))$ as follows:

1. $U^0 H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2)) = H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2))$
2. For $m \geq 1$, $U^m H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2))$ is the subgroup generated by $\delta(1 + \pi^m x) \cup \delta(y)$, where $\cup$ is the cup product of Equation (7.2.0.3), x is in $\mathcal{O}_{L^h}$, y is in $(L^h)^\times$, and δ is the connecting map $L^\times \to H^1(L, \mu_n)$ coming from Hilbert 90.

Theorem 7.3.0.1. *Let $e' := \frac{ep}{p-1}$, where $e = e(K/\mathbb{Q}_p)$. Assume that K contains a primitive p-th root of unity, so that in particular e' is an integer. Then*

$$\mathrm{fil}_{e'} \mathrm{Br}(L^h)[p] = \mathrm{Br}(L^h)[p],$$

$$U^{e'+1} H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2)) = (0),$$

and the isomorphism

$$\mathrm{Br}(L^h)[p] \cong H^2(L, \mathbb{Z}/p\mathbb{Z}(2))$$

induces, for each $0 \leq m \leq e'$, an isomorphism

$$\mathrm{fil}_{e'-m}(L^h)[p] \cong U^m H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2))$$

Definition 7.3.0.2. Let $Z^q_{k(Y)} = \ker(d : \Omega^q_{k(Y)} \to \Omega^{q+1}_{k(Y)})$ and $B^q_{k(Y)} = \mathrm{Im}\left(d : \Omega^{q-1}_{k(Y)} \to \Omega^q_{k(Y)}\right)$. Let moreover $\omega^q_{k(Y),\log}$ be the subgroup generated by differentials of the form $\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q}$, as the y_i vary over $k(Y)^\times$. Also define

$$\mathrm{gr}^m := \frac{U^m H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2))}{U^{m+1} H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2))}.$$

Proposition 7.3.0.3. *We have the following description of the graded pieces.*

1. *For $0 < m < e'$ with $p \nmid m$, there is an isomorphism*

$$\begin{array}{cccc} \rho_m : & \Omega^1_{k(Y)} & \xrightarrow{\sim} & \mathrm{gr}^m \\ & \overline{x}\frac{d\overline{y}}{y} & \mapsto & [\delta(1+\pi^m x) \cup \delta(y)], \end{array}$$

   where $x, y \in L^h$ are arbitrary lifts of $\overline{x}, \overline{y}$.

2. *For $0 < m < e'$ with $p \mid m$, there is an isomorphism*

$$\begin{array}{cccc} \rho_m : & \Omega^1_{k(Y)}/Z^1_{k(Y)} \oplus \Omega^0_{k(Y)}/Z^0_{k(Y)} & \xrightarrow{\sim} & \mathrm{gr}^m \\ & (\overline{x}\frac{d\overline{y}}{y}, \overline{z}) & \mapsto & [\delta(1+\pi^m x) \cup \delta(y)] + [\delta(1+\pi^m z) \cup \delta(\pi)], \end{array}$$

   where we have identified $B^2_{k(Y)} \oplus B^1_{k(Y)} \cong \Omega^1_{k(Y)}/Z^1_{k(Y)} \oplus \Omega^0_{k(Y)}/Z^0_{k(Y)}$.

3. *For $m = 0$, we have an isomorphism*

$$\begin{array}{cccc} \rho_0 : & \Omega^2_{k(Y),\log} \oplus \Omega^1_{k(Y),\log} & \xrightarrow{\sim} & \mathrm{gr}^0 \\ & (d\log\overline{y}_1 \wedge d\log\overline{y}_2, d\log\overline{y}) & \mapsto & [\delta(y_1) \cup \delta(y_2)] + [\delta(y) \cup \delta(\pi)]. \end{array}$$

Theorem 7.3.0.4. *Assume $\zeta_p \in L^h$. There exists a scalare $c \in k^\times$, depending on ζ_p and on π, such that*

1.
$$\mathrm{rsw}_{e',\pi}(\rho_0(\alpha, \beta)) = c(\alpha, \beta).$$

2. *If $p \nmid m$,*
$$\mathrm{rsw}_{e'-m,\pi}(\rho_m(\beta)) = c(d\beta, (e'-m)\beta)$$

3. *If $p \mid m$,*
$$\mathrm{rsw}_{e'-m,\pi}(\rho_m(\alpha, \beta)) = c(\alpha, \beta).$$

Remark 7.3.0.5. Recall that $U^m H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2)) \cong \mathrm{fil}_{e'-m} \mathrm{Br}(L^h)[p]$, and

$$\mathrm{rsw}_{n,\pi} : \frac{\mathrm{fil}_n \mathrm{Br}(L^h)[p]}{\mathrm{fil}_{n-1} \mathrm{Br}(L^h)[p]} \to \Omega^2_{k(Y)} \oplus \Omega^1_{k(Y)},$$

and $\frac{\mathrm{fil}_n \mathrm{Br}(L^h)[p]}{\mathrm{fil}_{n-1} \mathrm{Br}(L^h)[p]} \cong \mathrm{gr}^{e'-m}$.

Chapter 8

Brauer-Manin obstruction via refined Swan conductors IV

8.1 Recap

We let $K/\mathbb{Q}_p$ be a finite extension, $\mathcal{X}/\mathcal{O}_K$ be a smooth and proper variety, η_Y the generic point of $Y = \mathcal{X} \times_{\mathcal{O}_K} k$, $L^h = \mathrm{Frac}\left(\mathcal{O}^h_{\mathcal{X},\eta_Y}\right)$. We have defined two filtrations:

- Swan filtration: $\{\mathrm{fil}_n \mathrm{Br}(L^h)[p]\}_{n\geq 0} \subset \mathrm{Br}(L^h)[p]$. We also have maps $\mathrm{rsw}_{n,\pi} : \mathrm{fil}_n \mathrm{Br}(L^h)[p] \to \Omega^2_{k(Y)} \oplus \Omega^1_{k(Y)}$ such that $\ker(\mathrm{rsw}_{n,\pi}) = \mathrm{fil}_{n-1}$.
- Ramification filtration: $\{U^m H^2(H^h, \mathbb{Z}/p\mathbb{Z}(2))\}_{m\geq 0} \subseteq H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2))$.

  We have isomorphisms

$$\rho_0 : \Omega^2_{k(Y),\log} \oplus \Omega^1_{k(Y),\log} \xrightarrow{\sim} \mathrm{gr}^0$$

$$\rho_m : \Omega^1_{k(Y)} \xrightarrow{\sim} \mathrm{gr}^m \qquad \text{if } p \nmid m$$

$$\rho_m : B^2_{k(Y)} \oplus B^1_{k(Y)} \xrightarrow{\sim} \mathrm{gr}^m \qquad \text{if } p \mid m.$$

The groups $\mathrm{Br}(L^h)[p]$ and $H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2))$ are isomorphic if L^h contains a primitive p-th root of unity.

Theorem 8.1.0.1. *Let $e' = \frac{ep}{p-1}$.*

- $U^m H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2)) \cong \mathrm{fil}_{e'-m} \mathrm{Br}(L^h)[p]$
- $\mathrm{rsw}_{e'-m}(\rho_m(\beta)) = c(d\beta, (e'-m)\beta)$ *if* $p \nmid m$.
- $\mathrm{rsw}_{e'-m,\pi}(\rho_m(\alpha,\beta)) = c(\alpha,\beta)$ *for some* $c \in k^\times$ *if* $p \mid m$.

Remark 8.1.0.2. The theorem describes the image of $\mathrm{rsw}_{n,\pi}$ in $\Omega^2_{k(Y)} \oplus \Omega^1_{k(Y)}$.

Question 8.1.0.3. *What can we say about the image of*

$$\mathrm{rsw}_{n,\pi} : \frac{\mathrm{fil}_n \mathrm{Br}(X)[p]}{\mathrm{fil}_{n-1} \mathrm{Br}(X)[p]} \hookrightarrow H^0(\Omega^2_Y) \oplus H^0(\Omega^1_Y)?$$

8.2 Cartier operator

Let Y/k be a smooth variety. Write

$$Z^q_Y = \ker(d : \Omega^q_Y \to \Omega^{q+1}_Y)$$

and

$$B^q_Y = \mathrm{im}\left(d : \Omega^{q-1}_Y \to \Omega^q_Y\right).$$

Theorem 8.2.0.1. *There is an isomorphism*

$$C_Y^{-1} : \Omega_Y^q \to Z_Y^q / B_Y^q$$

that locally satisfies:

1. $C_Y^{-1}(\omega_1 \wedge \cdots \wedge \omega_q) = C_Y^{-1}(\omega_1) \wedge \cdots \wedge C_Y^{-1}(\omega_q)$;
2. *for* $q = 1$, $C_Y^{-1}(\lambda\omega) = \lambda^p C_Y^{-1}(\omega)$, *for all* $\lambda \in \mathcal{O}_Y$.

Definition 8.2.0.2. $\Omega^q_{Y,\log} = \ker(C_Y^{-1} - \mathrm{id} : \Omega_Y^q \to \Omega_Y^q / B_Y^q)$.

Remark 8.2.0.3.

- If $Y = \mathrm{Spec}(F)$, then $\Omega^q_{Y,\log} = \Omega^q_{F,\log}$ is the subgroup of Ω^q_F generated by $d\log y_1 \wedge \cdots \wedge d\log y_q$.
- The inverse of C_Y^{-1} is called the *Cartier operator* C. It fits in the short exact sequence

$$0 \to B_Y^q \to Z_Y^q \xrightarrow{C} \Omega_Y^q \to 0.$$

  (Note that C is only semilinear.) In particular, $\omega \in H^q(\Omega_Y^q)$ is in $H^0(B_Y^q)$ if and only if $d\omega = 0$ and $C\omega = 0$.

Corollary 8.2.0.4. *Let* $\mathcal{A} \in \mathrm{Br}(X)[p]$. *The following hold.*

1. $\mathrm{fil}_{e'} \mathrm{Br}(X)[p] = \mathrm{Br}(X)[p]$ *and* $\mathrm{rsw}_{e',\pi}(\mathcal{A}) = (\alpha, \beta)$ *with* $\alpha, \beta \in H^0(\Omega^2_{Y,\log}) \oplus H^0(\Omega^1_{Y,\log})$.
2. *If* $n < e'$ *and* $p \nmid n$, *then*
$$\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta) = (0,0) \Leftrightarrow \beta = 0.$$
3. *If* $n < e'$ *and* $p \mid n$, *then*
$$\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta) \in H^0(B_Y^2) \oplus H^0(B_Y^1)$$

Sketch of proof. 1. We have $\mathrm{fil}_{e'} \mathrm{Br}(L^h)[p] = \mathrm{Br}(L^h)[p]$, hence $\mathrm{fil}_{e'} \mathrm{Br}(X)[p] = \mathrm{Br}(X)[p]$.

2, 3. The inverse Cartier operator C_Y^{-1} fits in a commutative diagram

$$\begin{tikzcd} \Omega_Y^q \arrow[r, "C_Y^{-1}"] \arrow[d] & Z_Y^q / B_Y^q \arrow[d] \\ \Omega^q_{k(Y)} \arrow[r] & Z^q_{k(Y)} / B^q_{k(Y)}; \end{tikzcd}$$

both results follow combining this, the description of logarithmic forms as kernels, and the fact that $H^0(\Omega_Y^2) \oplus H^0(\Omega_Y^1)$ injects into $\Omega^2_{k(Y)} \oplus \Omega^1_{k(Y)}$.

□

8.3 p-adic vanishing cycles

Bloch & Kato generalise the definition of ramification filtration on $H^2(L^h, \mathbb{Z}/p\mathbb{Z}(2))$ to the sheaf

$$\mathcal{M}_2 := R^2 j_* \mathbb{Z}/p\mathbb{Z}(2),$$

where j is the inclusion of X into $\mathcal{X}$.

Fact 8.3.0.1.

1. *(Gabber) There is an injection*

$$H^0(\mathcal{X}, \mathcal{M}_2) \hookrightarrow H^0(L^h, H^2(L^h_{\mathrm{nr}}, \mathbb{Z}/p\mathbb{Z}(2))).$$

   The point here is that the right-hand side carries a natural ramification filtration.

2. *Using the second-page spectral sequence*

$$E_2^{s,t} = H^s_{ét}(\mathcal{X}, R^t j_* \mathbb{Z}/p\mathbb{Z}(2)) \Rightarrow H^{s+t}_{ét}(X, \mathbb{Z}/p\mathbb{Z}(2))$$

we get an edge map γ that fits in the diagram

$$\begin{tikzcd} H^2_{ét}(X, \mathbb{Z}/p\mathbb{Z}(2)) \arrow[r, "\gamma"] \arrow[d] & H^0(\mathcal{X}, R^2 j_* \mathbb{Z}/p\mathbb{Z}(2)) \arrow[d, hook, "g"] \\ H^2_{ét}(L^h, \mathbb{Z}/p\mathbb{Z}(2)) \arrow[r] & H^2_{ét}(L^h_{\mathrm{nr}}, \mathbb{Z}/p\mathbb{Z}(2))^{\mathrm{Gal}\left(L^h_{\mathrm{nr}}/L^h\right)}. \end{tikzcd}$$

In particular,

$$\ker \gamma = \{\mathcal{A} \in H^2_{ét}(X, \mathbb{Z}/p\mathbb{Z}(2)) \mid \mathcal{A}_{L^h} \in \ker\left(H^2_{ét}(L^h, \mathbb{Z}/p\mathbb{Z}(2)) \to H^2_{ét}(L^h_{\mathrm{nr}}, \mathbb{Z}/p\mathbb{Z}(2))\right)\}.$$

This group is very similar to how we defined fil_0 *on the Brauer group.*

Lemma 8.3.0.2. *The map γ induces a homomorphism*

$$\tilde{\gamma} : \mathrm{Br}(X)[p] \to H^0_{ét}(\mathcal{X}, \mathcal{M}_2)$$

such that $\ker \tilde{\gamma} = \mathrm{fil}_0 \mathrm{Br}(X)[p]$.

Proof. We have

$$\begin{tikzcd} H^2_{\text{ét}}(Z, \mathbb{Z}/p\mathbb{Z}(1)) \cong H^2_{\text{ét}}(X, \mathbb{Z}/p\mathbb{Z}(2)) \arrow[r] \arrow[d] & H^0(\mathcal{X}, \mathbb{Z}/p\mathbb{Z}) \\ \mathrm{Br}(X)[p] \arrow[ur, "\tilde{\gamma}"'] & \end{tikzcd}$$

To show the existence of $\tilde{\gamma}$ it suffices to prove $\gamma(\mathrm{Pic}(X)/p\,\mathrm{Pic}(X)) = 0$. Let $x \in \mathrm{Pic}(X)/p\,\mathrm{Pix}(X)$. By injectivity of g, we have $\gamma(x) = 0$ if and only if $g(\gamma(x)) = 0$. By Hilbert 90, we have $\mathrm{Pic}(L^j) = 0$, and commutativity of the diagram in the previous fact gives the statement. Finally,

$$\ker \tilde{\gamma} = \left\{\mathcal{A} \in \mathrm{Br}(X)[p] \mid \mathcal{A}_{L^h} \in \ker\left(\mathrm{Br}(L^h)[p] \to \mathrm{Br}(L^h_{\mathrm{nr}})\right)\right\}.$$

□

Corollary 8.3.0.3. *Let $\mathcal{A} \in \mathrm{Br}(X)[p]$ be such that $\tilde{\gamma}(\mathcal{A}) = 0$. There exists a finite unramified extension K'/K such that*

$$\mathrm{ev}_{\mathcal{A}} : X(K') \to \mathrm{Br}(K')$$

is non-constant.

Proof. If $\tilde{\gamma}(\mathcal{A}) \neq 0$, then $\mathcal{A}$ does not lie in $\mathrm{fil}_0 \mathrm{Br}(X)[p]$. Thus, there is some $n \geq 1$ such that $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta) \neq (0,0)$. If there exists $P_0 \in Y(k)$ such that $(\alpha_{P_0}, \beta_{P_0}) \neq (0,0)$, the evaluation map $X(K) \to \mathrm{Br}(K)$ is non-constant. The result follows from (i) one can always find a finite extension for which such a point P_0 exists; (ii) the refined Swan conductor rsw does not change under ramified extensions (i.e., $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = \mathrm{rsw}_{n,\pi}(\mathcal{A}_{K'})$). □

8.3.1 More on $H^0(\mathcal{X}, \mathcal{M}_2)$

From the Kummer sequence on $X_{\text{ét}}$,

$$1 \to \mathbb{Z}/p\mathbb{Z}(1) \to \mathbb{G}_m \xrightarrow{(\cdot)^p} \mathbb{G}_m \to 1$$

we get $j_*\mathbb{G}_m \xrightarrow{\delta} R^1 j_* \mathbb{Z}/p\mathbb{Z}(1)$. In analogy with what happens for fields, given $x, y \in j_*\mathbb{G}_m$ we get $\delta(x) \cup \delta(y) \in R^2 j_* \mathbb{Z}/p\mathbb{Z}(2)$.

Fact 8.3.1.1.

1. *$\mathcal{M}_2$ is generated (étale locally) by symbols of the form $\delta(x) \cup \delta(y)$ for $x, y \in j_*\mathcal{O}_X^\times$.*

2. These symbols can be used to define a ramification filtration

$$\{U^n\mathcal{M}_2\}_{n\geq 0},$$

given by

(a)
$$U^0\mathcal{M}_2 := \mathcal{M}_2.$$

(b)
$$U^n\mathcal{M}_2 = \langle \delta(1+\pi^m x)\cup\delta(y) : x\in\mathcal{O}_{\mathcal{X}}, y\in j_*\mathcal{O}_X^\times\rangle,$$

where by $\langle\cdots\rangle$ *we mean that the (sub)sheaf* $\mathcal{M}_2$ *is generated étale-locally by the given symbols.*

(c) *For* $m\geq e'$, $U^m\mathcal{M}_2 = \{0\}$.
For $0 < m < e'$ *with* $p\nmid n$, *there is an isomorphism* $\rho_m : \Omega^1_Y \to \mathrm{gr}^m(\mathcal{M}_2)$ *(note that the left-hand side is on the model and the right-hand side is on the special fibre).*
For $0 < m < e'$ *and* $p \mid m$, *there is an isomorphism* $\rho_m : B^2_Y \oplus B^1_Y \to \mathrm{gr}^m(\mathcal{M}_2)$.
Finally, for $m = 0$ *there is an isomorphism* $\rho_0 : \Omega^2_{Y,\log}\oplus\Omega^1_{Y,\log} \to \mathrm{gr}^0(\mathcal{M}_2)$.

Chapter 9

Brauer-Manin obstruction via refined Swan conductors V

9.1 Recall

$\mathcal{M}_2 = R^2 j_* \mathbb{Z}/p\mathbb{Z}(2)$ on $\mathcal{X}_{\text{ét}}$. There exists a filtration $\{U^m\mathcal{M}_2\}$ on $\mathcal{M}_2$ such that there exists an isomorphism

$$\begin{array}{cccc} \rho_0 : & \Omega^2_{Y,\log} \oplus \Omega^1_{Y,\log} & \to & \operatorname{gr}^0 \mathcal{M}_2 \\ & d\log \overline{y}_1 \wedge d\log \overline{y}_2 \mapsto \delta(y_1) \cup \delta(y_2) & & \end{array}$$

where y_1, y_2 are arbitrary lifts of $\overline{y}_1, \overline{y}_2$.

There is an exact sequence

$$0 \to U^1\mathcal{M}_2 \to \mathcal{M}_2 \xrightarrow{\rho_0^{-1}} \Omega^2_{Y,\log} \oplus \Omega^1_{Y,\log} \to 0.$$

Hence we have

$$\begin{tikzcd}
\operatorname{Br}(X)[p] \arrow[r, "\tilde{\gamma}"] \arrow[d] & H^0(\mathcal{X}, \mathcal{M}_2) \arrow[r, "\rho_0^{-1}"] \arrow[dd] & H^0(\Omega^2_{Y,\log}) \oplus H^0(\Omega^1_{Y,\log}) \arrow[dd] \\
\operatorname{Br}(K(X))[p] \ni (f,g) \arrow[d] & & \\
\operatorname{Br}(L^h)[p] \arrow[r] & \operatorname{Br}(L^h_{\text{nr}})[p])^{\text{Gal}} \arrow[r, "\rho_0^{-1}"'] & \Omega^2_{k(Y),\log} \oplus \Omega^1_{k(Y),\log}
\end{tikzcd}$$

Start from (f,g) in $\operatorname{Br}(L^h)[p]$. Then we get $\{f,g\}$ in $\operatorname{Br}(L^h_{\text{nr}})[p])^{\text{Gal}}$, which maps to $d\log \overline{f} \wedge d\log \overline{g}$ in $\Omega^2_{k(Y),\log} \oplus \Omega^1_{k(Y),\log}$, and now it's easy to lift this to (f,g) in $\operatorname{Br}(K(X))[p]$.

9.2 Ordinary varieties

Let Y/k be a smooth variety over a field of characteristic $p > 0$.

Definition 9.2.0.1. Y is called *ordinary* if $H^n(Y, B^q_Y) = 0$ for all n, q.

Remark 9.2.0.2.

- One can prove that for abelian varieties this is equivalent to the usual definition.
- Moreover, the definition is equivalent to

$$H^n(\Omega^q_{\overline{Y},\log}) \otimes_{\mathbb{F}_p} \overline{k} \cong H^n(\Omega^q_{\overline{Y}}).$$

- If Y is ordinary and such that $H^0(\Omega^2_Y) \neq 0$, then $H^0(\Omega^2_{Y,\log}) \neq 0$.

Example 9.2.0.3 (Ordinary abelian surface). Take $p = 3$, $K = \mathbb{Q}_3(\zeta_3)$, π a uniformiser, $k = \mathbb{F}_3$, and

$$\mathcal{E}/\mathcal{O}_K : y^2 = x^3 + 4x^2 + 3x + 1.$$

1. Let $\tilde{E}/\mathbb{F}_3$ be the special fibre, namely

$$\tilde{E}/\mathbb{F}_3 : y^2 = x^3 + x^2 + 1.$$

We claim that $\tilde{E}$ is ordinary. In particular, the global 1-form is given (locally) by the equation

$$\frac{dx}{2y} = -\frac{1}{2}\frac{(x-y)\,dx}{y(y-x)} = \frac{(x/y)\,dx - dx}{y-x} = \frac{dy-dx}{y-x} = d\log(y-x).$$

Let $\mathcal{S} := \mathcal{E} \times \mathcal{E}$, with respective coordinates $[X : Y : Z]$ and $[U : V : W]$. On the special fibre $\tilde{S}$, the global 2-form is given by

$$\omega := d\log(y-x) \wedge d\log(v-u).$$

Can we lift $\omega \in H^0(\Omega^2_{\xi,\log})$ to $\mathrm{Br}(S_K)[3]$? It's easy to do on the function field: note that $(y-x, u-v) \in \mathrm{Br}(K(S_K))[3]$.

We pause to give a criterion for an element of $\mathrm{Br}(K(X))[p]$ to extend to X.

Theorem 9.2.0.4. *Let $(f, g) \in \mathrm{Br}(K(X))[p]$.*

1. *Letting $D := \mathrm{Supp}(f) \cup \mathrm{Supp}(g)$ we have*

$$(f, g) \in \mathrm{Br}(U)[p], \text{ where } U = X \setminus D.$$

2. *We have an exact sequence*

$$0 \to \mathrm{Br}(X)[p] \to \mathrm{Br}(U)[p] \xrightarrow{\partial_{D_i}} \bigoplus_{\substack{D_i \leq D \\ \textit{irreducible}}} H^1(K(D_i), \mathbb{Z}/p\mathbb{Z}),$$

where, under the canonical identification $H^1(K(D_i), \mathbb{Z}/p\mathbb{Z}) \cong K(D_i)^\times/K(D_i)^{\times 2}$, the residue map ∂_{D_i} is

$$(f, g) \mapsto \left[(-1)^{\nu_{D_i}(f)\nu_{D_i}(g)} \frac{f^{\nu_{D_i}(g)}}{g^{\nu_{D_i}(f)}}\right].$$

Back to the example: let's apply the criterion given in the theorem. We have $D = \{Z = 0\} \cup \{X = Y\} \cup \{U = V\} \cup \{W = 0\}$. Let $U = S \setminus D$ and $(f, g) \in \mathrm{Br}(U)[3]$. Let for example $D_1 = (E \cap \{Z = 0\}) \times E = \{[0 : 1 : 0]\} \times E$, $D_2 = (E \cap \{X = Y\}) \times E = \{[1 : 1 : -1]\} \times E$.

We compute for example $\partial_{D_1}(f, g)$. Note that $\nu_{D_1}(g) = 0$, so that $\partial_{D_1}(f, g) = [g^{-\nu_{D_1}(f)}] \in K(D_1)^\times/K(D_1)^{\times 3}$. From the equation of E we obtain

$$Z(Y^2 - 4X^2 - 3XZ - Z^2) = X^3,$$

so – since $(Y^2 - 4X^2 - 3XZ - Z^2)$ does not vanish at $[0 : 1 : 0]$ – we get $\nu_{D_1}(f) = \nu_{D_1}((Y - X)/Z) = -\operatorname{ord}_{D_1}(Z) = -3\operatorname{ord}_{D_1}(X)$, which is divisible by 3. This shows that the residue is zero.

Exercise 9.2.0.5. *Check that $\partial_{D_2}(f, g) = 0$.*

Hint. Use that $Z(Y - X)(Y + X) = (X + Z)^3$ on E.

The upshot is that $\mathcal{A} = \left(\frac{Y-X}{Z}, \frac{V-U}{W}\right) \in \mathrm{Br}(S)[3]$ is such that $\tilde{\gamma}(\mathcal{A}) = (\omega, 0) \in H^0(\Omega^2_{Y,\log}) \oplus H^0(\Omega^1_{y,\log})$.

Theorem 9.2.0.6. *Assume that $\mathcal{X}/\mathcal{O}_K$ is such that*

1. *X_k is ordinary;*

2. *$H^i_{\mathrm{cris}}(Y/W(k))$ is torsion-free.*

Then, for every nonzero $\omega \in H^0(\Omega^2_Y)$, there exists a finite extension K'/K and $\mathcal{A} \in \mathrm{Br}(X_{K'})[p]$ such that $\mathcal{A}$ maps to $(\omega, 0)$ in $H^0(\Omega^2_Y) \oplus H^0(\Omega^1_{Y,\log})$.

9.3 K3 surfaces

Definition 9.3.0.1. K3 surfaces are 2-dimensional variety over K such that $H^0(\Omega^1_Y) = 0$ and $H^0(\Omega^3_Y)$ is a 1-dimensional vector space.

Fact 9.3.0.2.
- *If X/K is a K3 surface with good reduction, then the special fibre is also a K3 surface.*
- *For every homogenous $p(x, y, z, w)$, the zero locus $\{p = 0\} \subset \mathbb{P}^3_K$, if smooth, is a K3 surface.*
- *In this case, the global 2-form on X is given locally by*

$$\omega = \frac{d(y/x) \wedge d(z/x)}{\frac{1}{x^3}\partial p/\partial w}.$$

- *Y/k is ordinary if and only if $|Y(k)| \not\equiv 1 \pmod p$.*

Example 9.3.0.3 (An ordinary K3 for $p = 2$). Take $X/\mathbb{Q}_2$ given by

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0.$$

We have $|Y(\mathbb{F}_2)| = 10 \not\equiv 1 \pmod 2$, so Y is ordinary.

1. The global 2-form is (locally)

$$\omega = \frac{d(y/x) \wedge d(z/x)}{1/x^3(z^3 + w^2x + xyz)} = d\log\left(\frac{z^3 + w^2x + xyz}{x^3}\right) \wedge d\log\frac{z}{x}.$$

   Let $\overline{f} := \frac{z^3+w^2x+xyz}{x^3}$ and $\overline{g} := \frac{z}{x}$ (these are defined in characteristic 2; denote by f, g the obvious lift to $\mathbb{Q}_2$).

2. Does $(f, g) \in \mathrm{Br}(X)[2]$? The union of the supports of f, g consists of 5 irreducible divisors, including $\{x = y^3 + z^2w = 0\}$. We compute $\nu_{D_2}(f) = -3$, $\nu_{D_2}(g) = -1$, hence $\partial_{D_2}(f, g) = \left[-f^{-1}g^3\right] = [-1]$, which does not vanish in $\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2}$.

   Exercise 9.3.0.4. *$\partial_{D_i}(f, g) = 0$ for all $i \neq 2$.*

   The exercise implies that (f, g) belongs to $\mathrm{Br}(X_{\mathbb{Q}_2(i)})[2]$, and one can also check that $(f, -g) \in \mathrm{Br}(X)[2]$.

Remark 9.3.0.5. There are examples of K3 surfaces over $\mathbb{Q}_2$ with good ordinary reduction for which $\mathrm{Br}(X)[2] = \mathrm{fil}_0\,\mathrm{Br}(X)[2]$. In particular, in order to find an element with non-constant evaluation map we need a field extension of $\mathbb{Q}_2$.

Chapter 10

Brauer-Manin obstruction via refined Swan conductors VI

Goal. Do something similar to what we've seen yesterday (construct Brauer classes from global 2-forms) in the case of non-ordinary good reduction.

10.1 Recall

Let $\mathcal{X}/\mathcal{O}_K$ be smooth and proper, $K/\mathbb{Q}_p$ a finite extension, $j : X \hookrightarrow \mathcal{X}$ the inclusion of the generic fibre, and $\mathcal{M}_2 = R^2 j_* \mathbb{Z}/p\mathbb{Z}(2)$ the sheaf of 'p-adic vanishing cycles'. We have seen that there is a map

$$\tilde{\gamma} : \mathrm{Br}(X)[p] \to H^0(\mathcal{X}, \mathcal{M}_2)$$

such that $\ker(\tilde{\gamma}) = \mathrm{fil}_0 \mathrm{Br}(X)[p]$, and in particular, if $\mathcal{A} \notin \mathrm{fil}_0 \mathrm{Br}(X)[p]$, there exists a finite unramified extension K'/K such that $\mathrm{ev}_{\mathcal{A}} : X(K') \to \mathrm{Br}(K')$ is non-constant.

Remark 10.1.0.1. Assume that $\mathcal{A} \in \mathrm{Br}(X)[p]$ is such that there exists $n \geq 1$ with $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta) \in H^0(\Omega^2_Y) \oplus H^0(\Omega^1_Y)$ with $\alpha \neq 0$. For every finite extension K'/K we have

$$\mathrm{rsw}_{e(K'/K)n, \pi_{K'}}(\mathcal{A}_{K'}) = c \cdot (\alpha, e(K'/K)\beta)$$

for some non-zero constant $c \in k^\times$. This shows that the condition $\alpha \neq 0$ is stable under extensions of the ground field, while $\beta \neq 0$ is not. In particular, if $\mathcal{A}$ belongs to a certain stage of the filtration 'because of $\beta \neq 0$', when we pass to an extension the stage of the filtration can change. This doesn't happen if α doesn't vanish.

Corollary 10.1.0.2. *If* $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta)$ *with* $\alpha \neq 0$, *there exists a finite extension* K'/K *such that for all* $\tilde{K}/K'$ *the map* $\mathrm{ev}_{\mathcal{A}} : X(\tilde{K}) \to \mathrm{Br}(\tilde{K})$ *is non-constant.*

10.2 Non-ordinary case

Assume that $\mathcal{X}/\mathcal{O}_K$ is such that $H^0(\Omega^2_Y) \neq 0$. One can show that in many cases (more on this later; for abelian varieties, K3 surfaces, complete intersections, hyperKähler, ...) the following implication holds:

$$H^0(\Omega^2_{Y,\log}) = 0 \Rightarrow H^0(B^2_Y) \neq 0.$$

Recall that we have an isomorphism

$$\begin{array}{rccc} \rho_p : & B^2_Y \oplus B^1_Y & \to & \mathrm{gr}^p(\mathcal{M}_2) \\ & [\overline{f}\, d\log(\overline{g})] & \mapsto & [\delta(1+\pi^p f) \cup \delta(g)]. \end{array}$$

This induces a sequence

$$0 \to U^{p+1}\mathcal{M}_2 \to U^p \mathcal{M}_2 \to B^2_Y \oplus B^1_Y$$

and hence a map

$$H^0(\mathcal{X}, U^p\mathcal{M}_2) \to H^0(B^2_Y) \oplus H^0(B^1_Y).$$

Remark 10.2.0.1. A possible strategy to construct an 'interesting' class is to find $\mathcal{A} \in \mathrm{Br}(X)[p]$ such that

$$\tilde{\gamma}(\mathcal{A}) \in H^0(\mathcal{X}, U^p\mathcal{M}_2) \to H^0(B_Y^2) \oplus H^0(B_Y^1)$$

maps to a nonzero element in $H^0(B_Y^2) \oplus H^0(B_Y^1)$.

Remark 10.2.0.2. Recall that $U^m\mathcal{M}_2 = 0$ for $m \geq e'$. In particular, $U^p\mathcal{M}_2 \neq 0$ implies $p < e' := \frac{ep}{p-1}$, which works out to $e > p - 1$.

Example 10.2.0.3. Let $K = \mathbb{Q}_2(\sqrt{2})$ with uniformiser $\pi = \sqrt{2}$. We take X to be the K3 surface defined by

$$x^3y + y^3z + z^3w - w^4 + 2xyzw - \sqrt{2}xzw^2 = 0,$$

with special fibre

$$Y/\mathbb{F}_2 : x^3y + y^3z + z^3w + w^4 = 0.$$

We have $|Y(\mathbb{F}_2)| \equiv 1 \pmod 2$, so Y is not ordinary.

- Step 1: the 2-form on Y.

$$\begin{aligned} \omega = \frac{d(y/x) \wedge d(z/x)}{\frac{1}{x^3}\partial p/\partial w} &= \left(\frac{x}{z}\right)^2 d\left(\frac{y}{x}\right) \wedge d\log\left(\frac{z}{x}\right) \\ &= d\left(\frac{xy}{z^2}\right) \wedge d\log\left(\frac{z}{x}\right) \\ &= d\left(\frac{xy}{z^2}\, d\log\left(\frac{z}{x}\right)\right). \end{aligned}$$

  We thus take $\overline{f} = xy/z^2$ and $\overline{g} = z/x$.

- Step 2: compute $\rho_p^{-1}(\overline{f}\, d\log \overline{g}) = [\delta(1 + 2f) \cup \delta(g)$ with any lift of $\overline{f}, \overline{g}$ to $K(X)^\times$.

- Let $f_1 = 1 + 2f$, $g = z/x$ and take $\mathcal{A} = (f_1, g)$. By direct computations using

$$0 \to \mathrm{Br}(X)[2] \to \mathrm{Br}(U)[2] \to \bigoplus_{D_i \subseteq D \text{ irreducible}} H^1(K(D_i), \mathbb{Z}/2\mathbb{Z})$$

  and $D = \mathrm{Supp}(f) \cup \mathrm{Supp}(g)$, $U = X \setminus D$. We find that this has a non-trivial residue, but $(f_1, -g)$ has trivial residue along all D_i, so we have our class.

Example 10.2.0.4 (A family of examples). Take $K = \mathbb{Q}_2(\sqrt{d})$ with $d \in \mathbb{Z}$ a squarefree integer, $d \neq 0, 1$. Consider

$$X_d/K : x^3y + y^3z + z^3w - w^4 + dxyzw - \frac{2}{\sqrt{d}}xzw^2 = 0.$$

Since d is squarefree, $d \not\equiv 0 \pmod 4$, and hence $2/\sqrt{d}$ is in $\mathcal{O}_\mathfrak{p}$, where $\mathfrak{p}$ is the prime above 2. In particular, $\mathcal{X}/\mathcal{O}_K$ is smooth. There are two cases: if d is odd, the special fibre

$$Y_d : x^3y + y^3z + z^3w + w^4 + xyzw$$

is ordinary; if d is even, the special fibre is the same surface

$$x^3y + y^3z + z^3w + w^4$$

is zero.

The algebra $\mathcal{A}_d = \left(\frac{z^2+dxy}{z^2}, -\frac{z}{x}\right) \in \mathrm{Br}(X_d)[2]$ is such that

- for $d \equiv 1 \pmod 2$, $\tilde{\gamma}(\mathcal{A}_d) \in H^0(\mathcal{X}, \mathcal{M}_2)$ maps to $\omega_d \neq 0$ in $H^0(\Omega^2_{Y_d,\log})$;

- for $d \equiv 0 \pmod 2$, $\tilde{\gamma}(\mathcal{A}_d) \in H^0(\mathcal{X}, U^p\mathcal{M}_2)$ maps to $\omega_d \neq 0$ in $H^0(B_Y^2) \neq 0$.

10.3 A theorem from yesterday

Theorem 10.3.0.1. *Suppose Y/k is ordinary. Assume*

1. $H^0(\Omega^2_Y) \neq 0$
2. $H^i_{cris}(Y/W)$ *is torsion-free for all* i.

Then, for every $\omega \in H^0(\Omega^2_Y)$ there exists a finite extension K'/K and $\mathcal{A}\,\mathrm{Br}(X_{K'})[p]$ such that $\mathrm{ev}_{\mathcal{A}} : X(\tilde{K}) \to \mathrm{Br}(\tilde{K})$ for every finite extension $\tilde{K}/K'$.

Note of the scribe. I think different differential forms ω give different classes $\mathcal{A}$.

Theorem 10.3.0.2 (Ambrosi, Newton, Pagano)**.** *(Almost) the same holds without the ordinarity assumption (but not the 'for every ω'). More precisely: assume*

1. $H^0(\Omega^2_Y) \neq 0$
2. $H^i_{cris}(Y/W)$ *is torsion-free for all* i.

Then there exist a finite extension K'/K and $\mathcal{A}\,\mathrm{Br}(X_{K'})[p]$ such that $\mathrm{ev}_{\mathcal{A}} : X(\tilde{K}) \to \mathrm{Br}(\tilde{K})$ for every finite extension $\tilde{K}/K'$.

Proof. We have $\rho_0^{-1} : \mathcal{M}_2 \to \Omega^2_{Y,\log} \oplus \Omega^1_{Y,\log}$. Recall that $\rho_0 : \Omega^2_{Y,\log} \oplus \Omega^1_{Y,\log} \to \mathrm{gr}^0$ is an isomorphism. We get $(\rho_0^{-1})_{\overline{K}} : (\mathcal{M}_2)_{\overline{K}} \to \Omega^2_{Y_{\overline{k}},\log}$ and an exact sequence

$$0 \to (U\mathcal{M}_2)_{\overline{K}} \to (\mathcal{M}_2)_{\overline{K}} \to \Omega^2_{Y_{\overline{k}},\log} \to 0.$$

Fact 10.3.0.3 (Bloch-Kato)**.** *When Y is ordinary, $H^m((U\mathcal{M}_2)_{\overline{k}}) = 0$ for all n, and hence $H^0((\mathcal{M}_2)_{\overline{k}}) = H^0(\Omega^2_{Y_{\overline{k}},\log})$.*

Fact 10.3.0.4. *If $H^i_{cris}(Y/W)$ is torsion free, the spectral sequence of p-adic vanishing cycles degenerates, and the map*

$$H^2_{\acute{e}t}(X_{\overline{K}}, \mathbb{Z}/p\mathbb{Z}(2)) \xrightarrow{\gamma} H^0((\mathcal{M}_2)_{\overline{K}})$$

is surjective. This gives the existence of an $\mathcal{A} \in \mathrm{Br}(X_{\overline{K}})[p]$ such that $\tilde{\gamma}(\mathcal{A}_{\overline{K}}) \neq 0$.

□

10.4 Consequences over number fields

Let M be a number field and V be a smooth and proper variety over M. Suppose that for every finite extension M'/M we have

$$\overline{V(M')} = \prod_{v \in \Omega_{M'}} V(M_{v'}),$$

that is, V satisfies weak approximation over M'. Then $H^0(\Omega^2_V) \neq 0$.

Remark 10.4.0.1. This aligns with the expectation that varieties with many rational points should have a simple geometry.

To prove this statement, note that if there is a global 2-form, then Theorem 10.3.0.2 applies. More precisely:

Proof. For all but finitely many primes $\mathfrak{p}$ of $\mathcal{O}_M$, the variety $V_{\mathfrak{p}} = V \times_M M_{\mathfrak{p}}$ has good reduction, with special fibre $Y_{\mathfrak{p}}$, and $H^i_{cris}(Y_{\mathfrak{p}}/W(\kappa(\mathfrak{p})))$ is torsion-free. Assume that $H^0(\Omega^2_V) \neq 0$. Then for almost all primes of good reduction we have $H^0(\Omega^2_{Y_{\mathfrak{p}}}) \neq 0$. Theorem 10.3.0.2 gives the existence of a class $\mathcal{A} \in \mathrm{Br}((V_{\mathfrak{p}})_{K'})[p]$, where K' is a finite extension of $M_{\mathfrak{p}}$, such that $\mathrm{ev}_{\mathcal{A}}$ is non-constant for all finite extensions $\tilde{K}/K'$. Finally, one has to show that this class extends to a class in the Brauer group of the variety over a number field. This is true: since étale cohomology is invariant under base-change along inclusions of algebraically closed fields, we have

$$\mathcal{A}_{\overline{M}} \in \mathrm{Br}((V_p)_{\overline{M}_{\mathfrak{p}}}) \cong \mathrm{Br}(V_{\overline{M}}).$$

Since moreover $\mathrm{Br}(V_{\overline{M}}) = \bigcup_{M'/M} \mathrm{Br}(V_{M'})$, the class $\mathcal{A}$ is defined over some number field. □

Bibliography