POLARIZED ABELIAN VARIETIES FROM K-SECANTS (THE GENERAL STRATEGY).

1. RECOLLECTION

1.1. HODGE-WEIL CLASSES

X abelian variety over $\mathbb{C}$ of dimension $2n$.

Weil type $\phi : X \to X$ such that

- $\phi^2 = -d\,1_X$
- $\phi^* : H^{1,0}(X) \to H^{1,0}(X)$ has
  n eigenvalues $= i\sqrt{d}$ with sp.
  n '' values $= -i\sqrt{d}$ $_{\text{Im.}}$

$K := \mathbb{Q}(\sqrt{-d})$ $\qquad$ $(V_+ =)\ W \subseteq H^1(X)$ sub-space of eigenvalue $i\sqrt{d}$, $\dim(W) = 2n$

$H^1(X)$ is a K-vector space of $\dim\ 2n$

KENZA, RAPHAËL

$$W(X) := \underbrace{\Lambda_K^{2n} H^1(X)}_{\substack{\dim_K = 1 \\ \dim_{\mathbb{Q}} = 2.}} \subseteq H^{2n}(X) = \Lambda^{2n} H^1(X)$$

space of Weil classes and of type (n,n)

$$W(X) \otimes \mathbb{C} = \Lambda^{2n} W \oplus \Lambda^{2n} \bar{W}$$

algebra gen.

$$HW(X) := \langle W(X), c_1(\mathcal{L}) \rangle \subseteq H^{\bullet}(X) \text{ HODGE WEIL} \subseteq \mathrm{Hdg}^{2n}$$

$$\phi^*\mathcal{L} = d\mathcal{L}$$

$\mathrm{Pic}(X) \ni \mathcal{L}$ ample that gives a polarization "compatible"

WITH K

ANNA. IF Y IS VERY GENERAL $\check{H}W(Y)^{2i} = HOD^{2i}$ IS OF DIMENSION 1 IF $i \neq n$ AND 3 IF $i = n$.

AS AN ABELIAN VARIETY OF WEIL TYPE.

OSS IF Y DEFORMS, HW CLASSES DEFORMS IN HW CLASSES. IN PARTICULAR THEY STAY HODGE.

$H_1(X) \overset{\theta}{\xrightarrow{\sim}} H_1(X^\vee) \simeq H^1(X)$

1.2 K-SECANT AND ABELIAN VARIETIES.

LET X BE A PRINCIPALLY POLARIZED ABELIAN VARIETY

$\theta \in H^2(X) = \Lambda^2 H^1(X)$ POLARIZ. $\rightsquigarrow \theta : X \xrightarrow{\sim} X^\vee$

$Y := X \times X^\vee$ $\quad H^1(Y) = H^1(X) \oplus H^1(X)^\vee$ IS OF K-WEIL-TYPE WITH $\mathbb{Q}(\sqrt{-d})$

$\phi : Y \to Y \overset{p_1}{\to} X, \overset{p_2}{\to} X^\vee$ $\qquad \phi^2 = -d\, 1_Y$

$(x, \theta(y)) \longmapsto (-y, d\theta(x))$

BILINEAR FORM.

$c_1 : H^1(Y) \times H^1(Y) \to \mathbb{Q}$ $\qquad \psi : H^1(Y) \to \mathbb{Q}$

$((x,e),(x',e')) \longmapsto e(x') + e'(x)$ ASSOCIATED PLANAR FORM

$W = d p_1^* \theta + p_2^* \theta$ IS A POLARIZATION OF Y COMPATIBLE WITH K AND Y HAS DISCRIMINANT A POWER OF $d = N_{K/\mathbb{Q}}(\sqrt{-d})$

$\Rightarrow$ Y IS AN ABELIAN VARIETY OF SPLIT WEIL TYPE.

LEMMA. LET $W \subseteq H^1(Y)_\phi$ BE THE EIGENSPACE FOR $\sqrt{-d}$ FOR ϕ'. THEN W IS MAXIMAL ISOTROPIC (EQUIVALENT TO BE OF SPLIT WEIL TYPE)

RHAPAR $\Rightarrow$ EVERY ABELIAN VARIETY OF SPLIT WEIL TYPE DEFORM TO Y.

GUSEVA LAYLA

$$C(H^1(Y)) := \frac{T(H^1(Y))}{(v \otimes v - \frac{1}{2}q(v))} \supseteq C^+(H^1(Y)) \qquad cl(V) \xrightarrow[\sim]{m} End(H^\bullet)$$

GENERATED BY EVEN DEGREE TENSOR.

ACTS ON $H^\bullet(X)$ VIA (m_g)

HAS AN ANTI-INVOLUTION $(\cdot)^\circ$ AND

$$\mathrm{Spin}(H^1(X)) = \{x \in Cl^+(H^1(X)) \mid xx^\circ = 1,\ xH^1(X)x^{-1} \subseteq H^1(X)\}$$

ACTS ON $H^\bullet(X) = H^\bullet(X)^{even} \oplus H^\bullet(X)^{odd}$ VIA m_g FOR $g \in \mathrm{Spin}(H^1(X))$

__RECALL:__ $s \in H^\bullet(X)^{even}_{\mathbb{C}}$ IS A PURE SPINOR IF

PARENT IN GRADING $= -1$

$$\ker\left(H^1(X)_{\mathbb{C}} \to H^\bullet(X)^{-}_{\mathbb{C}},\ g \mapsto m_g(s)\right) =: K_s$$ IS MAXIMAL ISOTROPIC

AND FOR EVERY $W \subseteq H^1_{\mathbb{C}}(X)$ MAXIMAL ISOTROPIC $\exists!$ UP TO SCALAR $\varepsilon_W \in H^\bullet(X)^{even}$ SUCH THAT $W = K_{\varepsilon_W}$

$$\Rightarrow IG(2n,4n) \hookrightarrow \mathbb{P}(H^{even}(X))$$
$$W \longmapsto \varepsilon_W$$

__LEMMA.__ LET $W \subseteq H^1(X)_{\mathbb{C}}$ BE THE EIGENSPACE OF $i\sqrt{d}$ FOR ϕ'. THEN W IS MAXIMAL ISOTROPIC FOR $(\, , \,)$

⇝ W GIVES $\xi_{\bar{W}}, \xi_W \in \mathbb{P}(H^{even}(X))$ AND $P_X = \langle \xi_W, \xi_{\bar{W}} \rangle$

(DEFINED OVER k AND WHEN (NUCLEAR))

IS A LINE IN $\mathbb{P}(H^{even}(X))$ (k-)ISOTROPIC TO $\mathrm{IGr}(2m, 4m)$

OSS: ONE CAN SHOW THAT ALL THE ABELIAN VARIETIES OF SPLIT WEIL TYPE ARISE IN THIS WAY!!

$$\mathrm{Spin}(H^1(X))_{P_X} := \{ g \in \mathrm{Spin}(H^1(X)) \mid m_g(P) = 1 \ \forall P \in P_X \}$$

$\mathrm{Spin}(H^1(X))$ ACTS ALSO ON $H^\bullet(X)$

LEMMA: $(\Lambda^\bullet H(X)_{\mathbb{C}})^{\mathrm{Spin}(H^1(X))_{P_X}} = H^\bullet(X)^{\mathrm{Spin}}_{P_X}$

$$\begin{cases} 0 & \text{IN ODD DEG.} \\ \langle \theta^k \rangle & \text{IN DIM } 2k \neq 2n \\ \langle \Lambda^{2n} W, \Lambda^{2n} \bar{W}, w^n \rangle & \text{IN DIM. } 2n \end{cases}$$

COROLLARY: $HW(X) = H^\bullet(X)^{\mathrm{Spin}(H^1)_{P_X}}$

2. STRATEGY.

THM Let A be a 6-dimensional principally polarized abelian variety of split Weil type. Then the Weil classes are algebraic.

STRATEGY:

Fix an abelian principally polarized abelian variety X of dimension 3. Construct a coherent sheaf E of rank $r > 9$ on $X \times \hat{X} = Y$ such that

ⓐ E is semiregular

ⓑ $\alpha = ch(E) \cdot \exp(-c_1(E)/r) \in H^\bullet(Y)^{\mathrm{Spin}_\rho(V)}$

ⓒ α_{2n} and $cl(\theta)^n$ are independent

HOW THIS IMPLIES THE THM?

ANNA: WE CAN DEFORM A TO Y

(b) + COROLLARY $\Rightarrow \alpha = ch(E) \cdot \exp(-c_1(E)/r)$

$\in HW(Y)$

IN PARTICULAR, α STAYS HODGE IN EVERY DEFORMATION OF Y (AS P.A.V. OF ~~SPLIT~~ WEIL TYPE)

HENCE, (b) +MARKMAN E MATTEO $\Rightarrow \mathcal{E}$ DEFORMS TO A TWISTED COHERENT SHEAF ON EVERY SMALL DEFORMATION OF Y

$\Rightarrow \alpha$ REMAINS ALGEBRAIC ON EVERY DEFORMATION OF Y

$\Rightarrow$ THE DEFORMATION α_A OF α TO A IS AN ALGEBRAIC HODGE WEIL CLASS.

(c) $\Rightarrow \exists k \quad 1 \otimes \alpha_A - kh^3 \in HW(A)$. $\leftarrow$ TWO Q-DIMENSIONAL

$\Rightarrow \langle \alpha_A - kh^3 \rangle_K = HW(A)$

SINCE $\alpha_A - kh^n$ IS ALGEBRAIC AND $HW \subseteq HW(A)_{\mathbb{Q}}$ W/ ...

OBS 1 (WHY WE TWIST)

(a) THE INTEGRAL HODGE CONJECTURE FAILS FOR A GENERAL 6-FOLD.
THERE IS NO HOPE TO DEFORM WITHOUT TWIST.

(b) IN THE CONSTRUCTION OF $\alpha_{4} = 0$ BUT $c_{1}(E) \neq 0$. ONE CAN SHOW THAT $c_{1}(E) \notin H^{0}(X)^{\mathrm{sm}}$

OBS 2 (HODGE WNL VS WEIL).

FOR A VERY GENERAL COMPLEX TORUS OF DIM 5 W(X) ARE NOT ANALYTIC => W(X) DOES NOT DEFORM ANY FAMILY OF "ANALYTIC CYCLES" (THE POLARIZATION DOES NOT DEFORM?)

CANNOT EXIST A $\overset{\text{similar}}{\text{TWISTED}}$ SHEAF THAT WORKS FOR ~~A~~ WEIL CLASS.

OSS3 (WHY DIMENSION 3)

THE CONSTRUCTION OF $\mathcal{E}$ IS "BASED" ON THE FOLLOWING.

- $X = J(C)$ C GENUS 3 CURVE. $C \hookrightarrow J(C)$ (GEN.)

$\Rightarrow I_C$ IS A SEMI-MODULAR SHEAF ON X

THE SAME IS FALSE IF X HAS DIMENSION 4.

OSS4 (A DIFFICULTY)

THE SHEAF $\mathcal{E}$ IS NOT CONSTRUCTED DIRECTLY ON $X \times \hat{X}$ BUT ON A QUOTIENT $X \times \hat{X} / G$ FOR $G \subseteq X$ CYCLIC SUBGROUP. BEING SPIN EQUIVARIANT FOR $\frac{X \times \hat{X}}{G}$ MEANS THAT SPIN "ACTS AS G."

!