BOUNDARY COMPLEX OF M_g AND ITS CELLULAR HOMOLOGY

1. INTRODUCTION $g \in \mathbb{Z}_{\geq 2}$.

GOAL: EXPLICIT COMBINATORIAL DESCRIPTION OF

$$Gr^{W}_{6g-6} H^{6g-6-i}(M_g).$$

$$G_n = \left\{ (G, w) \;\middle|\; \begin{array}{l} \bullet\ G \text{ GENUS } g \text{ STABLE GRAPH WITH} \\ \quad - n+1 \text{ EDGES} \\ \quad - \text{NO LOOPS} \\ \quad - \text{NO POSITIVE WEIGHT VERTICES} \\ \bullet\ w : E[G] \simeq [n] \end{array} \right\} \Big/ \sim$$

$\sim$ ISOMORPHISM

$(G, w) \sim (G', w')$ IF

$\exists\ \sigma : G \cong G'$ GRAPH ISO SUCH THAT

$\sigma : E(G) \cong E(G')$

$w \downarrow \quad \downarrow w'$

$[n] = [n]$

$S_n \circlearrowright G_n$

$\sigma \cdot [G, w] = [G, \sigma \circ w]$

$K_n = \bigoplus_{G_n} [G, w) / \sim' \quad \left(= \left(\bigoplus [G, w) \otimes \mathrm{SIGN} \right)_{S_n} \right)$

$\sim' : (G, w) = \mathrm{SIG}[\sigma][G, \sigma \circ w'] \quad \sigma \in S_{n+1}$

$\partial_n : K_n \to K_{n-1}$

$$[G, w] \mapsto \sum (-1)^i [G/e_i, w|_{E(G)-e_i}]$$

<u>RMK</u> • IF G/e_i HAS A LOOP, WE PUT $G/e_i = 0$

<u>THM</u> (CGP)

$$\mathrm{GR}^W_{6g-6} H^{6g-6-i}(\mathcal{M}_g).$$
$$\shortparallel$$
$$\tilde{H}_{i-1}(K_\bullet)$$

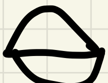
RMK $\pi : \mathrm{Aut}(G) \longmapsto \mathrm{Sym}(E(G)) \cong S_{n+1}$

$[G, \omega] = 0$ in K_n

$\Updownarrow$

$\exists\, \sigma \in \mathrm{Aut}(G)$ SUCH THAT $\mathrm{sign}(\pi(\sigma)) = -1$

$\left([G, \omega] = -[G, \sigma \cap \omega] = -[G, \omega] \right)$

EX. • $\Big[$  $\Big] = 0$

• IF G HAS TWO PARALLEL EDGES THEN $[G, \omega] = 0$

TWO STEPS

① G^h_n (FULL)

$$G^h_n = \{(G, w) \;\; G \text{ STABLE}, \; w : E(G) \cong [n]\}$$

$$\partial : G^h_n \longrightarrow G^h_{n-1}$$

$\Delta_g : I^{op} \longrightarrow SET$ SYMMETRIC Δ-COMPLEX.

$$[n] \longmapsto G^h_n$$

$$\begin{array}{ccc} [n] & & G^h_{n'} \\ \cap\, i & & \downarrow \\ [n'] & & G^h_n \end{array}$$

CONTRACT ALL THE EDGES IN $[n'] \setminus i([n])$

$K^h_\bullet := |K^h_n|$

<u>THM A</u> $|\Delta(M_g)| \simeq |\Delta_g|$

MATTIA $\Rightarrow GR^{w}_{6g-6} H^{6g-6-i}(M_g)$

$$\tilde{H}_{i-1}(K_\bullet^{h}) = \tilde{H}_{i-1}(|\Delta_g|, \mathbb{Q})$$

(2) $K_\bullet^{lw} \subseteq K_\bullet^{h}$

SUB COMPLEX MADE BY G WITH A LOOP OR A VERTEX OF POSITIVE WEIGHT.

BY DEFINITION.

$$\frac{K_\bullet^{h}}{K_\bullet^{lw}} \xrightarrow{\sim} K_\bullet$$

<u>THM</u> 2 $K_\bullet^{lw}$ IS ACYCLIC.

② THM 1

$\boxed{A}$ WHAT IS $|\Delta_g|$?

LEMMA $X : I^{op} \to SET$ Δ-COMPLEX.

$S_{n+1} \circlearrowright (n)$

$X(n) \to X(n-1)$

$[A] \mapsto [B]$

$\dfrac{X(n)}{S_n}$ SET OF ORBITS $= \{[A_1], \dots [A_m]\}$

$G_i = \mathrm{STAB}[A_i]$
$\cap$
S_{n+1}

$G_i \circlearrowright \Delta^n$

$\coprod \Delta^p$
$/\!/ X[p]$

$$S_n \circlearrowright \coprod_{p=0}^{\infty} X[p] \times \Delta^p \Big/ \sim$$

$x \in X[p]$, $s \in \Delta^{p-1}$, $d x \in X[p-1]$, $d s \in \Delta^p$

$\sim \qquad (x, \theta . \nu) \sim (\theta^{*} x, \nu)$

IF $\theta^{*} x = x$ THEN

$$\sigma \cdot \nu = \theta . \nu$$

<u>LEMMA</u> $|X| = \lim_{p} \left(\coprod_{A \in X[p]/S_{p+1}} \Delta^{p} / \mathrm{stab}(A) \right)$

$\Delta^{p-1}/\mathrm{stab}(A) \to \Delta^{p}/\mathrm{stab}(A')$ IF $\partial A' = A$

<u>IN OUR CASE</u>:

- AN S_n-ORBIT IS JUST A STABLE GRAPH OF GENUS g.
- $\mathrm{stab}([G, \sigma]) = \pi(\mathrm{Aut}(G)) \subseteq S_{n+1}$

$$S_{n+1} = \mathrm{Aut}(E(G))$$

SO $|\Delta_g| = \lim\limits_{n} \coprod \dfrac{\Delta^{E(G)}}{Aut(G)}$

GRAPH WITH n EDGES.

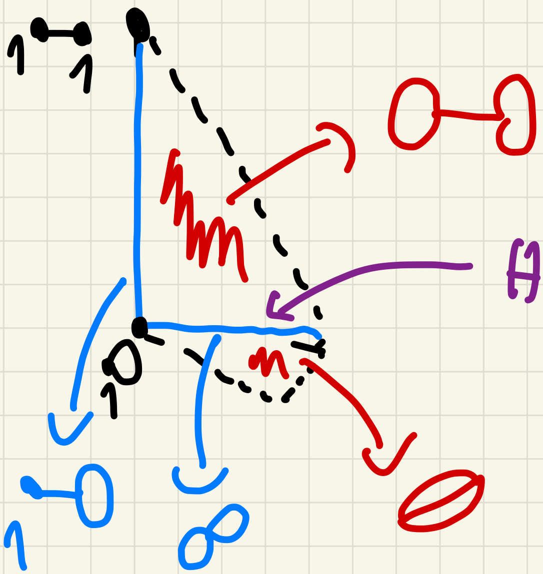
MODULI SPACE OF GRAPHS OF GENUS g.
+ LENGHT FUNCTION

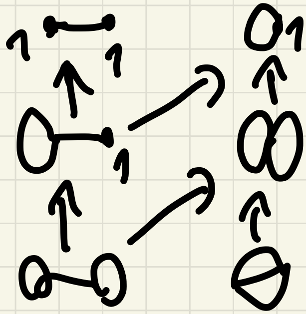
$$\ell : E(G) \longrightarrow \mathbb{R}_{\geq 0}$$

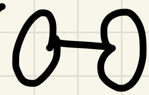
SUCH THA $\sum_{e \in E(G)} \ell(e) = 1$

1.B. TROPICAL CURVES

EX: Δ_2



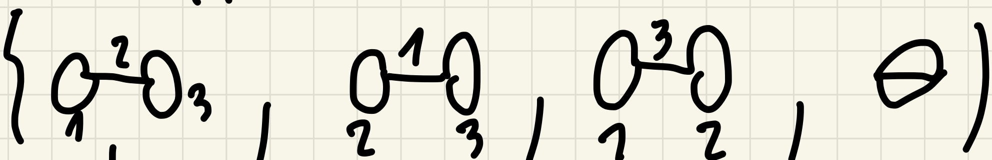


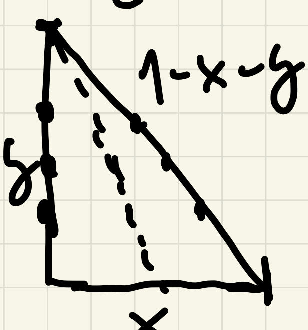
$\mathrm{Aut}($$) = \mathbb{Z}/2\mathbb{Z}$

$\mathrm{Aut}($$) = S_3$

$\Delta_2 \quad [2]$

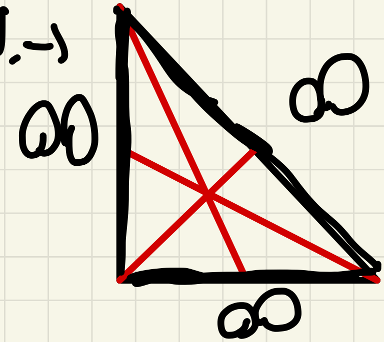
||





$$(x,y) \mapsto (x, 1-x-y) \qquad \mathbb{Z}/2\mathbb{Z}$$

MONDRIAN :-)



$$|\Delta(M_2)| \overset{\text{HOM}}{\simeq} \{\bullet\}$$

$$\Rightarrow \mathrm{GR}^{6}_{W}\left(H^\bullet(M_2)\right) = 0$$

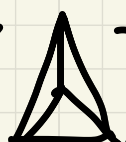
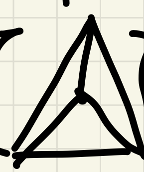
- $\Delta(M_3) \simeq ?$ $g = 3$

41 STABLE GRAPHS OF GENUS g

$$\mathbb{Q} = \mathrm{GR}^W_{12} H^6(M_g)$$

$K^{ch}_\bullet$ COMPLICATED

IN DEGREE 5 $\Uparrow$

$K_\bullet = \langle [$  $]_{w_3} \rangle$ AND $[$  $] \neq 0$

Q: WHAT IS $\Delta(M_g)$?

RECALL

(X, D) NORMAL CROSSING PAIR

(D_i) STRATIFICATION (OPEN)

$x_i \in D_i$

MONODROMY:

$\pi_1(D_i, x_i) \to \{$ BRANCHES THROUGH $x_i\}$

? LEMMA $|\Delta(X-D)| \simeq L(M \coprod \frac{\Delta^{Br(x_i)}}{\pi_1(M_i, x_i)})$?

com i strata

$\Delta^{Br(x_i)} \to \Delta^{Br(x_j)}$ IF $Br(x_i) \subseteq Br(x_j)$

$e_i \longrightarrow e_k$

TRANSITION MAP.

$Br(X_i) \subseteq Br(X_j)$ UP TO MONOMI[illegible]

IF $\bar{D}_j \subseteq \bar{D}_i$ BECAUSE

IF ONE CHOSE x_i NEAR x_j
$\cap$
D_i

THEN, LOCALLY, D_j 'S DEFINED

BY SOME ^MORE^ EQUATION TO ~~THE~~ ^OUR^ D_i ^OK^

ADDING.

RECALL

STABLE CURVES OF GENUS g

$$M_g \subseteq \overline{M}_g \supseteq \partial \overline{M}_g$$

⚠ $\partial \overline{M}_g$ IS NORMAL CROSSING BUT NOT SIMPLE NORMAL CROSSING.

- OPEN STRATA OF $\partial \overline{M}_g$ $\overset{1:1}{\longleftrightarrow}$ STABLE GRAPH OF GENUS g.

$$S \longmapsto G_S$$

CODIMENSION OF $S \longleftrightarrow |E(G_S)|$

$$S_1 \subseteq \overline{S}_2 \iff G_{S_2} \to G_{S_1}$$

EDGE CONTRACTION

COMBINATORIAL OF THE STRATA

CAN BE [illegible] OF TWO GRAPHS.

⚠ **PROBLEM** S MIGHT BE SINGULAR

THEN MIGHT BE MONODROMY.

RECALL $\prod_{v \in V(G)} M_{w(v),H(v)} = \tilde{M}_G$

$\downarrow$

M_G

ÉTALE GALOIS WITH GROUP $\mathrm{AUT}(G)$

$$M_G = [\tilde{M}_G / \mathrm{AUT}(G)]$$

MONODROMY

$$\pi_1(M_G, x) \curvearrowright \left\{ \begin{array}{c} \text{BRANCHES IN} \\ x \end{array} \right\} = E(G)$$

$$\left\{ \begin{array}{c} \text{BRANCHES IN} \\ x \end{array} \right\}$$

$$\|$$

$$\left\{ \begin{array}{l} \text{WAYS IN} \\ \text{WHICH } G \\ \text{CAN BE CONTINUED} \end{array} \right\} = E(G)$$

PROP

$$\begin{tikzcd} \pi_1(M_G, x) \arrow[r] \arrow[d, two heads] & \mathrm{SYM}(E(G)) \\ \mathrm{AUT}(G) \arrow[ur, "\pi"'] & \end{tikzcd}$$

SKETCH PROOF:

$$\prod_{v \in V(\Gamma)} M_{val(v),\, w(v)} \longrightarrow M_G$$

GALOIS-ÉTALE WITH GROUP $Aut(G)$

AND THE ACTION OF

$$\pi_1\left(\prod_{v \in V(\Gamma)} M_{val(v),\, w(v)} \right) \text{ IS TRIVIAL.}$$

WHY?

STACK AT A POINT IS THE

SET OF EDGES OF G

$\|$

NODES OF THE CURVES WITH

BECOME SMOOTH??

SO $$|\Delta(M_g)| = \varinjlim \coprod \frac{\Delta^{Bn(x_i)}}{\pi_1(M_i, x_i)}$$

$$\| $$

$$\varinjlim \coprod \frac{\Delta^{E(G)}}{Aut(G)}$$

$\boxed{C}$ CONCLUSION

LEMMA

$$|\Delta(M_g)| = \varinjlim \coprod \frac{\Delta^{Bn(x_i)}}{\pi_1(M_{G_i}, x_i)}$$

$\boxed{B}$ $\|$ ← PROP

$$\varinjlim \coprod \frac{\Delta^{E(G)}}{Aut(G)}$$

$$|\Delta_g| \qquad = \text{ ← PROP.}$$

$\|$ LEMMA →

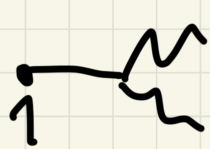
$$\varinjlim \coprod \Delta^{k} / Stab(G_i)$$

③ $\boxed{\text{THM 2}}$ $K^{wt}_\bullet$ is acyclic. EXTRA

($|\Delta^{wt}_g|$ is contractible).

I will prove that:

- $K^{w}_\bullet$ is acyclic.

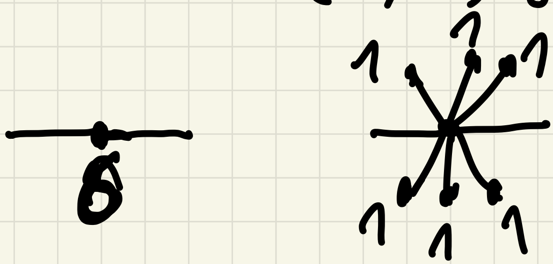
<u>DEF</u> A stem in G is a weight 1 vertex separated by an edge from the rest of G. (i.e. )

<u>CONSTRUCTION</u> Given a graph G with a vertex of positive weight let $G^s \to G$ be the maximal stem uncontraction of G.

- G^s HAS ONLY STEM AND
- $G^s \to G$ IF $\tilde{G} \to G$ CONTAINS ONLY STEM THEN

$$\begin{tikzcd} G^s \arrow[r] \arrow[dr, dotted, "\exists!"'] & G \\ & \tilde{G} \arrow[u] \end{tikzcd}$$

REPLACE $\forall v$ WITH $w(v)>0$ WITH $w(v)$ STEMS



LET $K_\bullet^{w,i-1} \subseteq K_\bullet^{w,i} \subseteq K_\bullet^{w,i+1} \subseteq K_\bullet^{w}$

SUBCOMPLEX OF GRAPHS WITH AT MOST i VERTICES THAT ARE NOT STEMS.

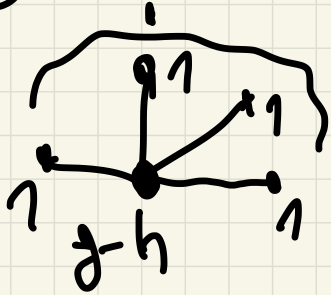
WE SHOW

(1) $K^{w,0}_\bullet$ IS ACYCLIC

(2) $\left(K^{w,i}_\bullet, K^{w,i-1}_\bullet\right)$ IS ACYCLIC.

(1) $K^{w,0}_\bullet$ $\forall \sigma$ HAVE ONLY STEMS

n EDGES



• IF $n \geq 2$ THEN

SO

$$[\sigma, \delta] = 0$$

$$K^{w,0}_\bullet = \left[K^w_1 \to K^w_0\right]$$

$$\shortparallel$$

$$\underset{g-1}{\bullet}\!\!-\!\!\underset{1}{\bullet} \longmapsto \underset{g}{\bullet}$$

$$\partial\left(\underset{g-1}{\bullet}\!\!-\!\!\!-\!\!\underset{1}{\bullet}\right) = \underset{g}{\bullet}$$

(2) $\dfrac{K_\bullet^{w,i}}{K_\bullet^{w,i-1}}$ GENERATED BY GRAPH WITH EXACTLY i NON-STEMS EDGES

$$K_\bullet^{w,i} \cong \bigoplus K_\bullet(G)$$ G SUCH THAT $G = G^S$

$$K_\bullet(G) = \bigoplus \mathbb{Q}[H, w] /\sim$$

H STEM CONTRACTION OF G

(CLEARLY $\langle K_\bullet(G)\rangle = K_\bullet^{w,i}$ AND $\bigcap K_\bullet(G) = 0$ BY $G = G^S$)

- WE ARE LEFT TO SHOW THAT

$K_\bullet(G)$ IS ACYCLIC.

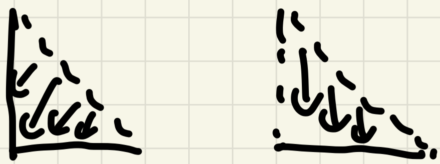
CLAIM

$K_\bullet(G)$ IS THE CHAIN COMPLEX $H_\bullet$ ASSOCIATED TO $\left(\Delta^{|E(G)|-1}/\mathrm{Aut}(G), Z/\mathrm{Aut}(G)\right)$

Z : FACES THAT CONTAINS VERTICES OF G THAT ARE STARS.

BUT $Z \subseteq \Delta^{|E(G)|-1}$ IS A DEFORMATION RETRACT OF $\Delta^{|E(G)|-1}$ IN A $\mathrm{Aut}(G)$-EQUIVARIANT WAY.

(E.G.



HENCE OK!

PROOF OF CLAIM

BY DEFINITION

$H_\bullet$ HAS A GENERATOR IN DIM p

$\forall$ $\mathrm{Aut}(G)$ ORBIT $[S, w]$

$S \subseteq E(G)$ IS A SET OF STEMS.

$$H_\bullet \longrightarrow K(G)$$

$$[S, w) \longrightarrow [G/S, w]$$

SURJECTIVE! BUT IT IS INJECTIVE:

IF $\quad [G/S, w] = [G/S', w']$

$$\begin{tikzcd} G \arrow[r] \arrow[d, dotted, "S"'] & G/S \arrow[d, phantom, "\cong"] \\ G \arrow[r] & G/S' \end{tikzcd}$$

SO $[S, w]$ $\shortparallel$ $[G/S, w]$ $\blacksquare$