DELIGNE – ILLUSIE ET APPLICATIONS

• INTRODUCTION

K CORPS PARFAIT, CAR$(K) = p$.

$Y \to X$ MORPHISME DE K-VARIÉTÉS.

EST RELEVABLE SUR $W_2(K) = W_2$ SI

$\exists \; Y_{W_2} \longrightarrow X_{W_2}$ (X, Y PLATS SUR W_2)

TQ.

$$\begin{array}{ccc} Y_{W_2} \times K & \longrightarrow & X_{W_2} \times K \\ \| & & \| \\ Y & \longrightarrow & X \end{array}$$

THM (P) $U \subseteq X$ IMMERSION OUVERTE DE K-VARIÉTÉS LISSES RELEVABLES SUR W_2.

X PROPRE. ALORS

$$\dim_K\left(\mathrm{Im}\left(H^m_{DR}(X) \to H^m_{DR}(U)\right)\right)$$
$$\geq$$
$$\dim_K\left(H^0(X, \Omega^m)\right)$$

SI $0 \leq m < p-1$

INTUITION :

LEMME :

$i : U \subseteq X$ VAR. LISSES SUR UN CORPS.

$$H^q(X, \Omega^i_X) \hookrightarrow H^q(U, \Omega^i_U)$$

PREUVE

IL SUFFIT $\Omega^i_X \hookrightarrow i_* \Omega^i_U$

ON PEUT SUPPOSER $X = \mathrm{Spec}(R)$ INTÈGRE
$U = \mathrm{Spec}(R_h)$ $h \in R - \{0\}$

$$\Omega^i_R \hookrightarrow \Omega^i_{R_h} = \Omega^i_R \otimes R_h$$

$R \hookrightarrow R_h$ ET Ω^i_R EST PLAT ∎

RAPPEL

THM (D-I) X PROPRE$/k$ RELEVABLE SUR W_2

ALORS $p \geq \dim(X)$ LA SUITE

$E_1^{i,j} = H^j(X, \Omega^i) \Rightarrow H^{i+j}_{DR}(X)$ EST DÉFINIE.
$\phantom{E_1^{i,j} = {}} \| $
$\phantom{E_1^{i,j} = {}} H^{i,j}(X)$

CELA NE SUFFIT PAS ! L'AFFINE

$$\begin{tikzcd} H^0(X, \Omega^i) \arrow[r, hook] \arrow[dr, hook] & H^i_{DR}(X) \arrow[d, hook] \\ & H^i_{DR} = \dfrac{\ker\left(H^0(U,\Omega^i) \xrightarrow{d} H^0(U,\Omega^{i+1})\right)}{\operatorname{Im}\left(H_U^{0,i-1} \xrightarrow{d} H_U^{0,i}\right)} \end{tikzcd}$$

U AFFINE

ÉTANT DONNÉE UNE FORME DIFF
EST-CE-QUE ON PEUT L'INTÉGRER
SUR X ?

LA QUESTION EST LOCALE, $V \subseteq \underset{\text{AFFINE}}{W} \subseteq X$

$$\begin{array}{ccccc}
H^q_W(\Omega^{i-1}) & \to & H^q_W(\Omega^i) & \to & H^{q,n+1}_W \\
\downarrow & & \downarrow & & \downarrow \\
H^q_V(\Omega^{i-1}) & \to & H^q_V(\Omega^i) & \to & H^q_V(\Omega^n)
\end{array}$$

$\underline{\text{EX}}$ $i = 2$ $\quad X = \mathbb{P}^1 \times \mathbb{P}^1$

$\mathrm{Pic}(X) = \mathbb{Z} \times \mathbb{Z}$

$C \subseteq X \rightsquigarrow (1,1) \in \mathrm{Pic}(X)$

$D \subseteq X \rightsquigarrow (1,2)$

$W = \mathbb{P}^1 \times \mathbb{P}^1 - C \quad \in \mathrm{Affine}$

$V = W - D \quad \in \mathrm{Affine}.$

tcm

$H^0(D) \to H^2_{dR}(X) = \mathbb{C}^2 = \mathbb{C}\langle C, D\rangle$

$\downarrow$

$H^2_{dR}(W) = \mathbb{C} = \mathbb{C}\langle D\rangle$

$H^2_{dR}(V) = 0$

- CAS COMPLEXE.

$U \subseteq X \leftarrow X$ PROPRE.

IMM. OUVERT VAR SUR $\mathbb{C}$.

$$\operatorname{Dim}_{\mathbb{C}}\left(\operatorname{Im}\left(H^n_{DR}(X) \to H^n_{DR}(U)\right)\right) \geq \operatorname{Dim}_{\mathbb{C}}\left(H^0(X, \Omega^n)\right)$$

PREUVE (SI $X-U$ EST LISSE)

SUITE D'EXCISSION + THÉORÈME DE HODGE MIXTE.

$$H^{n-2c}_{DR}(X-U)(-c) \to H^n_{DR}(X) \to H^n_{DR}(U)$$

SI T.H.

$$\bigoplus_{p\geq 0} H^{p,n-p}$$

$\bigoplus H$

$$H^{n-2c}(X-U) \cong \bigoplus_{p\geq 0} H^{p,n-2c-p}$$

SI

$$(-c) \quad \bigoplus_{p\geq 0} H^{p+2c,\, n-p}$$

$$\operatorname{im}\left(H^{n,s}_{x-v}\right) \subseteq H^{n,s}$$

$$p+2c>0 \qquad \operatorname{im} \cap H^{0,m} = 0$$

$p=m$ $\qquad H^{n+2c,0} = 0$

$$n+2c > m-2c.$$

$F_{X/K}$? DELIGNE - ILLUSIE.

$$\begin{tikzcd} X \arrow[d, "F_{X/K}"'] \arrow[dr, "F_X"] \arrow[dd, bend right] & \\ X^{(1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ K \arrow[r, "F_K"] & K \end{tikzcd}$$

Fini

CAR $K = p > 0$

X/K lisse (aussi ouverte)

Ex $\quad X = \mathbb{A}^1$

$$\sum a_i x^i \mapsto \sum a_i x^{ip}$$

$$\begin{tikzcd} K \arrow[r, "x \mapsto x^p"] \arrow[d] & K \arrow[d] \arrow[r] & K[x] \\ K[x] \arrow[r] \arrow[urr, bend left] & K[x] \arrow[ur] & \end{tikzcd}$$

$$\sum a_i x^i \longmapsto \sum a_i^p x^i$$

$$\sum a_i x_i^p \qquad \sum a_i x^i$$

$$F_{X/k,*}\Omega^\bullet_X := \left[F_{X/k,*}\mathcal{O}_X \xrightarrow{F_*d} F_{X/k,*}\Omega^1 \to F_*\Omega^2 \to \cdots\right]$$

MIRACLE : $F_{X/k,*}d$ EST $\mathcal{O}_{X'}$ LINÉAIRE.

$$d(x\cdot y^p) = d(x)y^p + \underbrace{p\,d(y^{p-1})y}_{} = 0$$

<u>THM</u> (CARTIER)

IL $\exists!$ HOMOMORPHISME D'ALGÈBRES GRADUÉES

$$C^{-1} : \bigoplus \Omega^i_{X^{(1)}/k} \xrightarrow{\sim} \bigoplus \mathcal{H}^i(F_*\Omega^\bullet_{X/k})$$

TEL QUE $C^{-1}(x\otimes 1) = x^{p-1}dx$

$$\mathcal{H}^i = \frac{\mathrm{Ker}(F_*\Omega^i_{X/k} \to F_*\Omega^{i+1})}{\mathrm{Im}(F_*\Omega^{i-1} \to F_*\Omega^i)}$$

PREUVE (SI $X = \mathbb{A}^1$)

$i=0$ $\quad k[x]$

$i=1$ $\quad k[x]\,dx$

$k[x]$ $x \mapsto x^p$

$$0 \to K[x^p] \to K[x] \xrightarrow{d} K[x]\, dx \to \bigoplus K \cdot x^{p^n-1} dx \to 0$$

$$K[x]\,dx, \quad dx \mapsto x^{p^n-1} dx$$

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para $i = 0$

$$\mathcal{O}_{X^{(1)}} = \Omega^0_{X^{(1)}/K} \cong \ker\left(F_*\mathcal{O}_X \to F_*\Omega^1_X\right)$$

para $i = 1$

$$\Omega^1_{X^{(1)}/K} \cong \frac{\ker\left(F_*\Omega^1_X \to \Omega^2_X\right)}{\operatorname{Im}\left(F_*\Omega^0 \to \Omega^1\right)}$$

THM (D-I) À $\tilde{X}/W_2(k)$ RELEVANT X EST ASSOCIÉ CANONIQUEMENT UN ISOMORPHISME

$$\varphi_{\tilde{X}} : \bigoplus_{i<p} \Omega^i_{X^{(1)}/k}[-i] \xrightarrow{\sim} \tau_{<p} F_* \Omega^\bullet_{X/k}$$

DANS $D(X^{(1)}, \mathcal{O}_{X^{(1)}})$

TEL QUE $\mathcal{H}^i(\varphi_{\tilde{X}}) = C^{-1}$, $i<p$

$$\Omega^i_{X^{(1)}/k} \to \mathcal{H}^i(F_{X/k\,*}\Omega^\bullet_{X/k})$$

$$M[-i]^j = M^{j-i}$$

$$d_{L[-i]} = (-1)^i d_L$$

$\tau_{<p} L^\bullet = \tau_{\leq p-1} L^\bullet$: L^i EN POS

$$(\tau_{\leq p-1} L^\bullet)^i = \begin{cases} L^i & i < p-1 \\ \ker(d) & i = p-1 \\ 0 & i > p-1 \end{cases}$$

$$L^i \to L^{i+1} \to \dots \quad L^{p-2} \to \ker(d) \to 0$$

$$H^i(\tau_{\leq p} L) = H^i(L) \quad i \leq p-1$$

$n = 1$

$$\mathrm{Ker}(F, \Omega^1 \to \Omega^2)$$

$$\downarrow$$

$$\Omega^1_{X^{(1)}/k} \underset{C^{-1}}{\simeq} \frac{\ker(\Omega^1_X \to \Omega^2_X)}{\mathrm{Im}(\Omega^0_X \to \Omega^{1}_X)}$$

D-I $\Rightarrow$ THM

$i : U \subseteq X \rightsquigarrow i : V_{W_2} \subseteq X_{W_2}$

$$\text{D-I} \Rightarrow \bigoplus_{i<p} \Omega^i_{X^{(1)}/k}[-i] \to i_* \bigoplus_{i<p} \Omega^i_{V^{(1)}/k}[-i]$$

$$\downarrow \wr \qquad\qquad\qquad \downarrow \wr$$

$$\tau_{<p} F_{X/k} \Omega^\bullet_{X/k} \to i_* \tau_{<p} F_{X/k} \Omega^\bullet_{X/k}$$

$i \leq p-1$

$$\begin{tikzcd}
H^i(X^{(1)}, \Omega^i_{X^{(1)}/k}) \arrow[r, hook, "\text{Lemma 8}"'] \arrow[d, equal] & H^i(V^{(1)}, \Omega^i_{X^{(1)}/k}) \\
H^i(X^{(1)}, \bigoplus_{i<p} \Omega^i_{X^{(1)}/k}[-i]) \arrow[r] \arrow[d, "\cong"] & H^i(V^{(1)}, \bigoplus \Omega^i_{X^{(1)}/k}[-i]) \arrow[d, "\cong"] \\
H^i(X^{(1)}, \tau_{<p} F_{X/k,*}\Omega^\bullet_{X/k}) \arrow[r] \arrow[d, "\cong"] & H^i(V^{(1)}, \dots) \arrow[d, "\cong"] \\
H^i(X^{(1)}, F_{X/k,*}\Omega^\bullet_{X/k}) \arrow[r] \arrow[d, "\cong"] & H^i(\ , \dots) \arrow[d, "\cong"] \\
H^i(X, \Omega^\bullet_{X/k}) \arrow[r] \arrow[d, equal] & {} \arrow[d] \\
H^i_{dR}(X) \arrow[r] & H^i_{dR}(V)
\end{tikzcd}$$

③ PREUVE DE DELIGNE ILLUSIE.

X/W AFFINE.

POUR $n=1$ ET F RELEVABLE :

$$\begin{tikzcd} X \arrow[rrd, bend left, "F_X"] \arrow[rd, "F_{X/K}"'] \arrow[rdd, bend right] & & \\ & X^{(1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ & K \arrow[r, "F_K"'] & K \end{tikzcd}$$

$$\begin{tikzcd} \tilde X \arrow[r, "\exists \tilde F_{X/K}"] \arrow[rd] & \tilde X^{(1)} \arrow[r] \arrow[d] & \tilde X \arrow[d] \\ & W_2 \arrow[r, "F"'] & W_2 \end{tikzcd}$$

ON A

$$\begin{tikzcd} X \arrow[r] \arrow[d, "F_{X/K}"'] \arrow[dr, phantom, "\square"] & \tilde X \arrow[d, "\tilde F"] \\ X^{(1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde X^{(1)/K} \arrow[d] \\ K \arrow[r] & W_2 \end{tikzcd}$$

(LOCALEMENT VRAI)

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on veut :

$$\Omega^1_{X^{(1)}/K} \xrightarrow{\ \sim\ } \operatorname{Ker}(F_*\Omega^1 \to F_*\Omega^2)$$

$$\Omega^1_{X^{(1)}/K} \overset{?}{\dashrightarrow} F_*\Omega^1_X \hookleftarrow \tilde F_*\Omega^1_{\tilde X} \hookleftarrow p\tilde F_*\Omega^1_{\tilde X} = 0$$

$$F_*\Omega^1_X \xrightarrow[\sim]{p} p\tilde F_*\Omega^1_{\tilde X}$$

① $F^*\Omega^1_{X^{(1)}/K} \xrightarrow{0} F_*\Omega^1_{X/K}$

$k[x]dx \longrightarrow k[x]dx$

$x \longmapsto dx^p = 0$

$\to pW_2(k) \to W_2(k)$

- $\mathrm{im}(\tilde F) \subseteq p\Omega^1_{X/k}$
- $\tilde F(p\Omega^1_{\tilde X^{(1)}/k}) \subseteq p\,\mathrm{im}(\tilde F) \subseteq p^2\Omega^1_{X/k} = 0$

$\tilde F(x) = x^p + ph$

$$d(\tilde F(x)) = d(x^p + ph)$$

$$\begin{aligned} &\quad\quad \| \\ &d(x^p) + ph \\ &\rightsquigarrow p\,x^{p-1}dx + p\,dh \end{aligned}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & p\Omega^1_{\tilde X^{(1)}/k} \arrow[r, "0"] \arrow[d] & p\Omega^1_{X/k} \arrow[r] \arrow[d] & 0 \\
 \tilde F: & \Omega^1_{\tilde X^{(1)}/k} \arrow[r] \arrow[d] \arrow[ur, red] & F_*\Omega^1_{X/k} \arrow[d] & \\
 F: \dfrac{\Omega^1_{\tilde X/k}}{p} = & \Omega^1_{X^{(1)}/k} \arrow[r, "0"] \arrow[d] \arrow[uur, blue] & F_*\Omega^1_{X/k} \arrow[d] \arrow[uu, bend right, "p"'] & \\
 & 0 & 0 &
\end{tikzcd}$$

$dx \mapsto dx^p$

$f := p^{-1}\tilde{F}$. est-ce que $-\, df = 0$?

$-\, \mathcal{X}^1(h) = C^1$

$$\tilde{F}(x \oplus 1) = x^p + p\,h$$

$$\boxed{h(dx_0 \oplus 1) = x_0^{p-1}\,dx_0 + dh(x)}$$

$$\Downarrow$$

$$p\,d(x_0 \oplus 1) = \tilde{F}(dx \oplus 1) = p\,x_0^{p-1} + p\,d(h)$$

$$\shortparallel$$

$$p\,x_0^{p-1}\,dx + p\,d(h)$$

$h = C^1$

$$df(dx_0 \oplus 1) = 0$$

$$\Updownarrow$$

$$d\,p^{-1}\tilde{F}(dx_0 \oplus 1) = 0$$

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{F}_p \arrow[r] & \dfrac{\mathbb{F}_p[x]}{x^2} \arrow[r] & \mathbb{F}_p \arrow[ll, bend right, "x"', "\sim"]
\end{tikzcd}
$$

$$K[x,y]\,dx \oplus K[x,y]\,dy$$

$$\downarrow$$

$$K[x,y]\,dx \wedge dy$$

$$h(x,y)\,dx + g(x,y)\,dy$$

$$\downarrow d$$

$$\left(\frac{dg(x,y)}{dx} - \frac{df(x,y)}{dy}\right) x \wedge y$$

$$g(x,y) = 0$$

$$df = \frac{\partial f(x,y)}{\partial x} dx + \frac{f(x,y)}{\partial y} dy$$