COHOMOLOGIE MOD p DES SCHÉMAS FORMELS.

1) ÉNONCÉS

NOT k PARFAIT DE CAR $p>0$

$$\begin{array}{ccc} W(k) & , & K = \mathrm{FRAC}(W(k)) \\ \cap & / & \cap \\ \mathcal{O}_{\mathbb{C}_p} & \subseteq & \mathbb{C}_p = \widehat{\overline{K}} \end{array}$$

$$\begin{tikzcd} Y_0 \arrow[r, "f_0"] \arrow[d] \arrow[rr, bend left] & X_0 \arrow[d] & Y \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ k \arrow[rrr] & & & W(k) \end{tikzcd}$$

SCHÉMAS FORMELS LISSES

X PROPRE.

$$\begin{tikzcd} Y_\eta \arrow[r] \arrow[dr] & X_\eta \arrow[d] \\ & \mathbb{C}_p \end{tikzcd}$$

ESPACE RIGIDE SUR $\mathbb{C}_p$

SI X PROPRE. $X_\eta = X^{AN}_{\mathbb{C}_p}$

THM! (K-F-W)

$$\dim_{\mathbb{F}_p}\Big(\underbrace{\mathrm{Im}\big(H^i_{ET}(X_\eta, \mathbb{F}_p) \xrightarrow{f^*} H^i(Y_\eta, \mathbb{F}_p)\big)}_{\mathrm{Im}^i_{ET}}\Big)$$

$$\geq \dim_k\Big(\underbrace{\mathrm{Im}\big(H^i_{dR}(X_0) \xrightarrow{f_0^*} H^i_{dR}(Y_0)\big)}_{\mathrm{Im}^i_{DR}}\Big)$$

SI $p > n+2$ ($p \geq n+3$) $\quad (i < p-2 \iff p-2 > i)$

COR $Y \hookrightarrow X$ IMMERSION OUVERTE $\quad p > i+2$ ($p \geq n+3$)

$$\dim\left(\mathrm{Im}^i_{\text{ét}}\right) \geq \dim\left(H^0\left(X_K, \Omega^i_{X_K}\right)\right)$$

PREUVE

$$\dim_{\mathbb{F}_p}\left(\mathrm{Im}^i_{\text{ét}}\right) \overset{\text{THM}}{\geq} \dim_k\left(\mathrm{Im}^i_{\text{dR}}\right)$$

$p > i$, $p-1 \geq i$

VI $\in$ D-I $\quad (p > n+1)$

$$H^0\left(X_K, \Omega^1_K\right) \leq \dim_k\left(H^0\left(X_k, \Omega^i_{X_k}\right)\right)$$

SEMI-CONTINUITÉ.

EXPOSÉ 1. $\cancel{X}/W_2$

VARIANTE $Y \hookrightarrow X$ OUVERT, $p > 3$ $\quad p \geq i+2$, $n < \dim(X)$

(1) $$\dim\left(\mathrm{Im}\left(\Lambda^i H^1_{\text{ét}}(X_{\bar{\eta}}) \to H^i(Y_{\bar{\eta}})\right)\right) \geq \binom{\dim(H^0(X_K, \Omega^1_M))}{i}$$

② X SCHÉMA AMÉLIORÉ. $p \geq i+2$

$i \leq p-2$

$i+2 \leq p$

STRATÉGIE : UTILISER LA COHOMOLOGIE PRISMATIQUE.

2§ RAPPELS SUR LA COHOMOLOGIE PRISMATIQUE

$\mathfrak{Z} \to W(k)$ SCHÉMA FORMEL LISSE.

DEUX PRISMES • $\Delta_{BK} := \left(W(k)[[u]], \varphi : u \mapsto u^p\right)$

$\hat{A}_{inf} = W(\mathbb{C}_p)$

$A_{inf} := W(\mathcal{O}_{C^\flat})$ • $\Delta_{inf} := \left(A_{inf}, \varphi\right)$

$\mathcal{O}_{C^\flat} = \lim\limits_{x \mapsto x^p} \mathcal{O}_C/p = \widehat{\overline{k[[u]]}}$

$$\begin{tikzcd}
u \arrow[d, mapsto] & W(k)[[u]] \arrow[r, hook, "u \mapsto {[}p^\flat{]}"] \arrow[d, "\theta_{BK}"] & A_{inf} \arrow[r] \arrow[d, "\theta_{inf}"] & A_{inf}\left[\frac{1}{\xi}\right] \\
p & W(k) \arrow[r] & \mathcal{O}_{C_p} &
\end{tikzcd}$$

θ_{inf} ← THÊTA

$\ker(\theta_{BK}) = (u-p)$ $\qquad$ $\ker(\theta_{inf}) = (p - [p^\flat]) = (\xi)$

$\mathfrak{m} \ni p \quad [p] \quad p - [\underline{p}]$

$\mathfrak{m} \quad \mathfrak{m} \quad \underline{p} = (p, p^{\frac{1}{p}}, p^{\frac{1}{p^2}}, \dots)$

$$\begin{array}{ccccc} & & k[[m]] & \longrightarrow & \mathcal{O}^\flat_{\mathbb{C}_p} = k[[\underline{m}]] \longrightarrow A_{INF}[\tfrac{1}{\xi}] = k((m))^{\Delta} \\ & & \downarrow & & \downarrow \\ \mathbb{F}_p & \longrightarrow & k & \longrightarrow & \mathcal{O}_{\mathbb{C}} \end{array}$$

$H^i \subset \overline{H^\bullet(Z, \Delta_{BK})} = H^\bullet(Z, \Delta_{BK}) \overset{L}{\otimes} \mathbb{F}_p / k[[m]]$

$H^i \in \overline{H^\bullet(Z, \Delta_{INF})} = H^\bullet(Z, \Delta_{INF}) \overset{L}{\otimes} \mathbb{F}_p / \mathcal{O}^\flat_{\mathbb{C}_p}$

THM

DR: $\overline{H^\bullet(Z, \Delta_{BK})} \overset{L}{\otimes}_{k[[m]]} k \xrightarrow{\sim} H^\bullet_{DR}(Z_k)$ (1)

ÉT: $\overline{H^\bullet(Z, \Delta_{INF})} \otimes A[\tfrac{1}{\xi}] \xrightarrow[\xi = 0]{\sim} H^\bullet_{ET}(Z_{\mathbb{C}})$ (2)

BC: $\overline{H^\bullet(Z, \Delta_{BK})} \otimes A \cong \overline{H^\bullet(Z, \Delta_{INF})}$ (3)

$VF = p^i$: $\varphi^* \overline{H^\bullet(Z, \Delta_{BK})} \xrightarrow{F} \overline{H^\bullet(Z, \Delta_{BK})}$ (5)

$$\overline{H^i(Z, \Delta_{BK})} \xrightarrow{v} \varphi^* \overline{H^i(Z, \Delta_{BK})}$$

$$\mathrm{Fav} = n^i = \mathrm{VOF}$$

- SI Z PROPRE. $\overline{H^i(Z)}$ SONT FIN GEN.

CONCRÈTEMENT:

$f : R \to S$ ANNEAUX $\qquad C^\cdot$ COMPLEXE SUR R.

$$E_2^{pq} := \mathrm{TOR}_R^p(H^{-q}(C^\cdot), S) \Rightarrow H^{-q-p}(C^\cdot \otimes^L S)$$

- SI f PLAT. $\quad H^i(C^\cdot \otimes^L S) = H^i(C^\cdot \otimes S)$

- SI $\mathrm{Tor}^i(H^{-q}(C^\cdot), S) = 0 \quad i \geq 2$

$$0 \to H^i(C) \otimes S \to H^i(C^\cdot \otimes^L S) \to \mathrm{Tor}_R(H^{i+1}(C^\cdot), S) \to 0$$

$\boxed{\text{EX}}$ $k[[u]]$ P.I.D. $\qquad \mathrm{Tor}^i = 0 \ \forall\, i \geq 2$.

$$\overline{H^\cdot_{BK}(Z)} \otimes^L k \simeq \overline{H^\cdot_{BK}(Z_k)}$$

$$0 \to H^i_{qk}(Z)\otimes R \to H^i_{m}(Z_R) \to \mathrm{Tor}^1_R\left(\overline{H^{i+1}_{\Delta_{qk}}(Z_R)}, R\right)$$

$$H^{i+1}_{\Delta_{qk}}(Z_R)[n] \hookrightarrow 0$$

- $H^i_{qk}(Z)^{\otimes R} \hookrightarrow H^i_{m}(Z_K)$
- $H^i_{qk}(Z) = H^i_{m}$ si $\overline{H^{i+1}_{\Delta_{qk}}(Z)}$ sans torsion.

$\boxed{8}$ $\overline{H^{i-1}_{\Delta_{INF}}(Z)} \to H^i_{\acute{E}T}(Z) \to \overline{H^i_{\Delta_{INF}}(Z)} \to \overline{H^{i+1}_{\Delta_{INF}}(Z)}$

<u>DONC</u>

$$H^i_{\acute{E}T}(Z)\otimes\mathbb{C}^b \longrightarrow \overline{H^i_{\Delta_{INF}}(Z)}\otimes\mathbb{C}^b$$

$$H^i_{\acute{E}T}(Z)\otimes\mathbb{C}^b \simeq \overline{H^i_{\Delta_{INF}}(Z)}\otimes\mathbb{C}^b \quad Z \text{ propre.}$$

3) ÉCHAUFFEMENT.

<u>LEMME</u> $Z \to W(k)$ PROPRE.

$$\dim\left(H^i_{DR}(Z_k)\right) \geq \dim\left(H^i_{\acute{e}t}(Z_{\bar{k}})\right).$$

$=$ SI $H^i_{\Delta_{BK}}(Z)$ ET $H^{i+1}_{\Delta_{BK}}(Z)$ N'ONT PAS DE TORSION.

<u>RMQ</u> POURQUOI C'EST CONC?

$X/\mathbb{Z}_2$ ENRIQUES E.C. $X \in K3$. $/\mathbb{Z}/2\mathbb{Z}$

$$H^1_{DR}(X_{\mathbb{C}}) = 0$$
$$H^1_{\acute{E}T}(X_{\mathbb{C}}) = \mathbb{Z}/2\mathbb{Z}$$

$$H^1_{DR}(X_k) = \mathbb{Z}/2\mathbb{Z}.$$

<u>PREUVE</u>

$$\dim(H^i_{\acute{E}T}(Z_{\bar{k}})) \overset{\acute{E}T}{=} \dim(H^i_{\acute{E}T}(Z) \otimes \mathbb{C}^b)$$

$\parallel \Leftarrow Z$ PROPRE.

$$\dim\left(H^i_{\Delta ENF}(Z)\otimes k^{D}\right)$$

$/\!/ \in B - i$

$$\dim\left(H^i_{\Delta dR}(Z)\otimes k((u))\right)$$

Z propre $\to \leq$ || si $H^i_{dR}(Z)(u)=0$

$$\dim\left(H^i_{\Delta dR}(Z)\otimes k\right)$$

$\leq$ || $H^{i+1}_{dR}(Z)[u]=0$

$$\dim\left(H^i_{dR}(Z)\right)$$

$$0\to H^i_{dR}(Z)\otimes k \subseteq H^i_{dR}(Z_k) \to Tm^1\left(H^{i+1}_{dR} / k\right) \simeq 0$$

PROBLÈMES DANS LE CAS RELATIF :

① $g : \overline{H^i_{dR}(X)} \to \overline{H^i_{dR}(Y)}$

$$\dim\left(\mathrm{Im}(g)\otimes k((u))\right) \underbrace{\leq} \dim\left(\mathrm{Im}(g)\otimes k\right) \overset{\dim}{\geq} \mathrm{Im}(g\otimes k)$$

ON A BESOIN D'ÉGALITÉ.

$\mathrm{Im}(g)$ SANS TORSION

(2)

$$\begin{tikzcd}
H^i_{cr}(X|\otimes k) \arrow[r, "?"] \arrow[d] & H^i_{cr}(Y|\otimes k) \arrow[d] \\
H^i_{DR}(X_n) \otimes k \arrow[r] \arrow[d] & H^i_{DR}(Y_n) \\
\mathrm{Tm}(H^{i+1} \quad) &
\end{tikzcd}$$

RMQ : $\mathrm{Im}(H^i_{A_{inf}} \to H^i_{A_{inf}})$

DE PRÉSENTATION FINIE ?

OUI CAR

$$\mathrm{Im}(H^i_{A_K} \xrightarrow{} H^i_{A_K}) \otimes K[\bar{\pi}].$$

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§4 RÉSULTAT DE "SANS-TORSIONITÉ"

RÉSULTAT DE BASE :

<u>LEMME</u> R : $\mathbb{F}_p$-ALGÈBRE. $\varphi_R : R \to R$ FROB. ABSOLU

M A MODULE DE PRÉSENTATION FINIE.

SI $\varphi^* M \simeq M \Rightarrow M$ EST PLAT.

<u>PREUVE POUR $R = k[[x]]$</u>

$$M \simeq \bigoplus R^n \oplus \bigoplus_i \frac{k[[x]]}{x^{m_i}}$$

$$\frac{k[[x]]}{x^{m_i}} \otimes_{\varphi} \frac{k[[y]][x]}{(y^p - x)} \simeq \frac{k[[y]]}{y^{pm_i}}$$

ET
$$\bigoplus \frac{k[x]}{(x^{m_i})} \not\cong \bigoplus \frac{k[x]}{(x^{pm_i})}$$

EX PERMUTATION FINIE EST ESSENTIEL.

$\frac{k[[x]]}{k[[x]]}$. [IL SUFFIT TORSION BORNÉE]

ON N'A PAS $F \circ V = m^i$ PARTOUT. DONC?

LEMME : M FIN. GEN. $k[[m]]$-MODULE.

$F: \sigma^* M \to M$ $\quad V: M \to \sigma^* M$

TELS QUE $F \circ V = V \circ F = m^i$.

SI $p \geq d + 2$ M EST SANS-TORSION.

HYPOTHÈSE SUR d MINIMAL.

$d + 1 < p$
$p > d + 1$

$M = k$ $\quad \sigma^* M = \frac{k[m]}{(m^p)}$

$\dot{d} = p$

$$M \xrightarrow{0} f^! M \xrightarrow{0} M$$

$\dot{d} = p+1$

$$M \to f^! M \to M$$
$$1 \longmapsto u^{p-1}$$
$$1 \longmapsto 1$$

<u>PREUVE</u> n minimale tel que.

$$u^n x = 0 \quad \forall x \in M$$

$$\exists x \quad u^{n-1} x \neq 0$$

$$M \hookrightarrow f^! M = M \otimes_{3} k(u^?)$$
$$m \longmapsto (m \otimes 1)$$

⟵ la plus petite puissance de u qui tue $x \otimes 1$ est u^{pn}

$$u^{i+n}(x \otimes 1) = u^i(x \otimes u^n) = V^i F(x \otimes u^n)$$

$$u^{pn} F(x \otimes 1) = F(x \otimes u^{pn}) = F(u^n x \otimes 1) \quad V(u^n F(x \otimes 1))$$

$F(x \otimes 1)$ est de torsion

$\overset{i}{d+m \geq pm}$ $\qquad$ $\overset{i}{d} \geq \cancel{p}m(p-1) \geq p-1$ $\qquad$ 0

so $d < p-1$ $\qquad$ $p-1 > d$ $\qquad$ $p \geq d+2$

SANS FACSION.

5 § Pour authénomies.

$p \geq n+3$

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & X_0 \arrow[d] \\ k \arrow[r] & \mathrm{Frac}(K) \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[dl] \\ \mathrm{Frac}(K) \arrow[r] & \mathbb{C}_p \end{tikzcd} \qquad \begin{tikzcd} Y_\eta \arrow[r] \arrow[d] & X_\eta \arrow[dl] \\ \mathbb{C}_p & \end{tikzcd}$$

$$\mathrm{Dim}(\mathrm{Im}^i_{\bar{e}t}\,\eta) = \mathrm{Dim}(\mathrm{Im}^i_{dR}) \ ?$$

$$\mathrm{Dim}\left(\mathrm{Im}\,^i_{\bar{E}T}(X) \longrightarrow \mathrm{Im}\,^i_{\bar{E}T}(Y)\right)$$

$$\|$$

$$\mathrm{Dim}\left(\overline{H^i_{\bar{E}T}(X) \otimes \mathbb{C}_p^b} \to \overline{H^i_{\bar{E}T}(Y) \otimes \mathbb{C}_p^b}\right).$$

PAS ÉGAL $\longrightarrow$ $\not\|$

$$\begin{tikzcd} H^i_{\bar{E}T}(X) \otimes \mathbb{C} \arrow[r] \arrow[d, "\wr"] & H^i_{\bar{E}T}(Y) \otimes \mathbb{C} \\ H^i_{INF}(X) \arrow[r] & H^i_{INF}(Y) \otimes \mathbb{C}^b \end{tikzcd}$$

$$\mathrm{Dim}\left(H^i_{\Delta\,INF}(X) \otimes \mathbb{C}^b\right) \to H^i_{INF}(X) \otimes \mathbb{C}^b)$$

$$\|_{CB}$$

$$\mathrm{Dim}\left(H^i_{\Delta_{B,K}}(X) \otimes K((t)) \to H^i_{\Delta_{B,K}}(Y) \otimes K((t))\right)$$

VI $\leftarrow$ $p \geq n+2$

$$\mathrm{Dim}\left(H^i_{\Delta_{BK}}(X) \otimes k \longrightarrow H^i_{\Delta_{BK}}(Y) \otimes k\right)$$

$\mathrm{Im}(f)$ sans torsion!
On applique le comp. à $\mathrm{Im}\left(H^i_{\Delta_{BK}}(X)\right) \supseteq H^i_{\Delta_{BK}}(Y)$

$$\mathrm{Dim}\left(\mathrm{Im}(\cancel{f\otimes k})\right) = \mathrm{Dim}(\mathrm{Im}(f)) = \mathrm{Dim}(\mathrm{Im}(f) \otimes k)$$

$$\geq \mathrm{Dim}(\mathrm{Im}(f \otimes k))$$

VI $p \geq n+3$

$$\mathrm{Dim}\left(H^i_{dR}(X_0) \longrightarrow H^i_{dR}(Y_0)\right)$$

$$\begin{array}{ccc}
H^i_{dR}(X_0) & \longrightarrow & H^i_{dR}(Y_0) \\
\downarrow & & \downarrow \\
H^i_{BK}(X_0) \otimes k & \longrightarrow & H^i_{dR}(Y_0) \otimes k \\
\wr\!\!\mid & & \downarrow \\
H^i_{dR}(\quad) & \longrightarrow & H^i_{dR}(Y_0) \\
\downarrow & &
\end{array}$$

$\mathrm{Tor}^{n}(H^{n+1}_{m}(\,|\,))$ $\quad p \geq i+q+2 \geq i+3$

$n+1$ SANS TORSION

SI. $H^{n+1}_{BK}(x_n)|$. SANS TOR. IL SUFFIT

$p \geq n+2$.