① SEMINORMAZIONE PER FIBRATI/FASCI COERENTI

1.1 CLASSI DI CHERN:

$$c : \left\{ \begin{array}{c} \text{FIBRATI} \\ \text{VETTORIALI} \\ \text{SU } X \end{array} \right\} / \sim \; \longrightarrow \; \bigoplus H^{2i}(X, \Omega^i) \subset \bigoplus H^{2i}(X)$$

CHERN CLASSES

- ASSIOMI: $c = \sum c_k$, $c_k \in H^i(X, \Omega^i)$

- $c_0(E) = 1$
- $c(f^*E) = f^* c(E)$
- $c(E \oplus F) = c(E) \cup c(F)$
- L LINE BUNDLE, $c(L) = 1 + c_1(L)$

- X, $E \to X$ FIBRATO

$H^\bullet(\mathbb{P}(E))$ ALGEBRA LIBERA CON UN GENERATORE ξ SU $H^\bullet(X)$ (DI RANGO $r = \mathrm{RANK}(E)$)

$$\xi^r - \pi^*(c_1(E))\xi^{r-1} + \dots + (-1)^r \pi^*(c_r(E)) = 0$$

ES $c_0 = 1$, $c_1 = cl(\det(E))$

CH(E) CHERN CHARACTER

$\in H^\bullet$

$CH(E)_0 = RANK(E)$

$$CH(E)_k = \frac{1}{k!} \, DET \begin{pmatrix} c_1 & 1 & 0 & 0 & 0 & 0 \\ 2c_2 & c_1 & 1 & & & \\ \vdots & c_2 & \ddots & \ddots & & \\ \vdots & & \ddots & c_2 & c_1 & 1 \\ kc_k & c_{k-1} & \cdots & \cdots & & c_1 \end{pmatrix}$$

$k > 0$

ESSE $E = \bigoplus L_i$ $\qquad \sum_i \exp(c_1(L_i))$

$$\sum_i \sum_j \frac{c_1(L_i)^j}{j!}$$

$CH(E)_1 = c_1$

$$CH(E)_2 = \frac{1}{2} \, DET \begin{pmatrix} c_1 & 1 \\ 2c_2 & c_1 \end{pmatrix} = \frac{1}{2} \left(c_1^2 - 2c_2 \right)$$

$$CH(E)_3 = \frac{1}{6} \begin{pmatrix} c_1 & 1 & 0 \\ 2c_2 & c_1 & 1 \\ 3c_3 & c_2 & c_1 \end{pmatrix} = \frac{1}{6} \left(c_1^3 - 3c_1c_2 + 3c_3 \right)$$

$CH(E \otimes \mathcal{L}) = CH(E) \cdot CH(\mathcal{L})$

OSS c_k È UN POLINOMIO RAZIONALE IN

ch E c_i $i<k$.

SE CONOSCO ch E c_1 POSSO CALCOLARE c_i $\forall i$

TEO $\mathrm{Vect}(X) \xrightarrow{ch} H^{2\bullet}$

$CH^\bullet(X) \xrightarrow{cl} H^{2\bullet}$

GENERANO (ASTRATTAMENTE) COTTORA LA STESSA IMMAGINE!

"DEFORMARE CICLI" = "DEFORMARE FIBRATI"

1.2 CLASSI ATIYAH E SEMIREGOLARITÀ.

A ALGEBRA / $\mathbb{C}$: $0 \to I \to A \otimes_{\mathbb{C}} A \xrightarrow{m} A \to 0$

$$0 \to \frac{I}{I^2} \to \frac{A\otimes A}{I^2} \to \frac{A \otimes A}{I} \cong A \to 0$$

$$\frac{I}{I^2} \cong \Omega^1_A$$

$$\Omega^1_A \to \frac{I}{I^2} \qquad \frac{I}{I^2} \to \Omega^1_A$$

$$b_1 \otimes b_2 \to b_1 d b_2$$

$$da \longrightarrow (1\otimes a - a \otimes 1)$$

$$0 \to \Omega^1_A \xrightarrow{i} \frac{A\otimes A}{I^2} \to A \to 0 \qquad \text{SU } \frac{A\otimes A}{I^2} \text{ o } A\otimes A$$

VEDIAMO ORA X LISCIA.

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "i_*"'] & X \times X \\ \mathcal{X} \arrow[ur] & \end{tikzcd}$$

$\mathcal{X}$ = DEFINITA DA I^2

$$p_1^* f \otimes i_* g = i_*(i^* p_1^* f \otimes g) = i_*(f \otimes g)$$

$$0 \to i_* \Omega^1_X \to \mathcal{O}_\mathcal{X} \to i_* \mathcal{O}_X \to 0 \qquad \mathcal{E} \text{ PIATTO}$$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[rrr, bend left, "1_X"] & \mathcal{X} \arrow[r, hook] & X \times X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \\ & & X & \end{tikzcd}$$

$$0 \to i_*\Omega^1_X \otimes \pi_1^*\mathcal{E} \to \pi_1^*\mathcal{E} \to i_*\mathcal{O}_X \otimes_{\pi_{2,*}\mathcal{O}_X} \overset{\pi_2}{\mathcal{M}} \to 0$$

$$i_*\Omega^1_X \otimes \pi_1^*\mathcal{E} = i_*(\Omega^1 \otimes \mathcal{E})$$

$$0 \to \pi_{1,*} i_* \Omega^1_X \to \pi_{1,*}\mathcal{O}_\mathcal{X} \to \pi_{1,*} i_* \mathcal{O}_X \to 0$$

$$0 \to \Omega^1_X \to \pi_{1,*}\mathcal{O}_\mathcal{X} \to \mathcal{O}_X \to 0$$

$\mathrm{Hom}(\mathcal{E}, - \otimes \mathcal{E}) = - \otimes \mathcal{E} \otimes \mathcal{E}^\vee$

$\tilde{e} \mapsto (m \otimes e) \in (m, e, f)$

$$0 \to \Omega^1_X \otimes \mathcal{E} \to \pi_{1,*}\mathcal{O}_\mathcal{X} \otimes \mathcal{E} \to \mathcal{E} \to 0 \qquad (\phi_\mathcal{E})$$

$\to \mathrm{Hom}_D(\mathcal{E}, \Omega^1 \otimes \mathcal{E}[1])$

<u>DEF</u>: $At_\mathcal{E} = [\phi_\mathcal{E}] \in \mathrm{Ext}^1(\mathcal{E}, \Omega^1_X \otimes \mathcal{E}) = H^1(X, \Omega^1_X \otimes \mathrm{End}(\mathcal{E}))$

E CLASSE DI ATIYA DI E

$$H^1(X,\Omega^1\otimes END(E)) \xrightarrow{TR} H^1(X,\Omega^1)$$

LEMMA IF E_1 IS A LINE BUNDLE $TR([AT_E]) = c_1(E)$

$$\begin{tikzcd}
H^1(X,\Omega^1\otimes END(E)) \times H^1(X,\Omega^1\otimes END(E)) \arrow[d] \arrow[dd, bend right, "\cup"'] \\
H^2(X,(\Omega^1)^{\otimes 2}\otimes END(E)^{\otimes 2}) \arrow[d] \\
H^2(X,\Omega^2\otimes END(E))
\end{tikzcd}$$

$$[AT_E]^2 = AT_E \cup AT_E$$

DEF $$[AT_E]^n = \underbrace{AT_E \cup \dots \cup AT_E}_{n\text{-VOLTE}} \in H^n(X,\Omega^n\otimes END(E))$$

LEMMA $TR([AT_E]^n) = \frac{1}{k!} c_k(E)$

D^b

$[\mathrm{Hom}(M,N[i])$

$\mathrm{Ext}^i(M,N)$

$At^n_E : E \to E \otimes \Omega^n[n]$ IN D^b

$\sigma_n : Ext^2(E,E) \xrightarrow{At^n} Ext^{n+2}(E, \Omega^n \otimes E) \xrightarrow{tr} H^{n+2}(\Omega^n)$

DEF. LA MAPPA DI SEMIREGOLARITÀ È

$$\sigma : Ext^2(E,E) \xrightarrow{\prod \sigma_n} \prod_{q \geq 0}^{d-2} H^{q+2}(M, \Omega^q)$$

• E È SEMIREGOLARE SE σ È INIETTIVA.

ES SIA $\mathcal{L}$ FIBRATO IN RETTE. ALLORA $n=0$

$Ext^2(L,L) = H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X) \xrightarrow[\sim]{tr} H^2(\mathcal{O}_X)$

QUINDI $\mathcal{L}$ SEMIREGOLARE.

THM (BUCHWEITZ-FLENNER) $Y \to X$ LISCIA PROIETTIVA, E_x / Y_x FIBRATO SEMIREGOLARE.

ASSUMI $c_\bullet(E) \in H^\bullet(Y_x)^{\pi_1(X)}$ ALLORA

$\exists U \subseteq X \mid E|_U \mid E|_{Y_x} = E_x$. IN PARTICOLARE

$\gamma(C.C(E))$ È ALGEBRICA $\forall \gamma$ (DIMOSTRATO = A QUELLA DI BLOCH)

THM 1 (MARKMAN) $Y \to X$ VAR. ABELIANA (NON NO)

E_x FIBRATO SEMIREGOLARE $\mid ch(E_x)\cdot ch(\frac{1}{r}DET(E_x)) \in H^\bullet(Y_x)^{TAU}$

ALLORA $\forall \gamma$ $\gamma(ch(E)\, ch(\frac{1}{r}DET(E)))$ È ALGEBRICA $\forall \gamma$.

FASCI TWISTATI

(CAPITOLO 2016 PERRY IN GENERALE)

X $\mathcal{U}=\{U_i\}$ BUON RICOPRIMENTO ($U_{i_1}\cap \dots \cap U_{i_j}$ SEMPLICEMENTE CONNESSO)

$U_{ij} = U_i \cap U_j$ $\quad U_{ijk} = U_i \cap U_j \cap U_k$

L FIBRATO IN RETTE $\longleftrightarrow H^1(X, \mathcal{O}_X^*)$

$$L \longrightarrow \{ g_{ij} : L_{|U_{ij}} \xrightarrow{\sim} L_{|U_{ji}} \} = \{ L_{ij} \in \mathcal{O}^*_{U_{ij}} \mid g_{ii}=1,\ g_{ij}=g_{ji}^{-1},\ g_{ij}\circ g_{jk}=g_{ik} \}$$

$$0 \to \mu_n \to \mathcal{O}_X^* \xrightarrow{n} \mathcal{O}_X^* \to 0$$

$$0 \to \frac{Pic(X)}{n Pic(X)} \xrightarrow{\delta_n} H^2(X,\mu_n) \to H^2(X,\mathcal{O}_X^*)[n]$$

$\mu_n \cong \mathbb{Z}/n\mathbb{Z}$

• $H^2(X, \mu_n)$ e $H^2(X, \mathcal{O}_X^*)$

$$\alpha_{i,j,k} \in \mathcal{O}^*_{U_{i,j,k}} \quad | \quad (\alpha_{i,j,k}^n = 1)$$

$$\alpha_{ijk} = \alpha_{ijl} \cdot \alpha_{ikl}^{-1} \cdot \alpha_{jkl}$$

<u>ES</u> $\mathcal{L}$ FIBRATO IN RETTE

$$\delta_n(\{g_{i,j}\}) = \{h_{ij}\, h_{jk}\, \overset{\alpha_{i,j,k}}{\overset{\shortparallel}{h_{ik}^{-1}}}\}$$

DOVE h_{ij} È RADICE n-ESIMA DI $g_{i,j}$ SU $U_{i,j,k}$

V FIBRATO VETTORIALE $\longleftrightarrow H^1(X, GL_n)$

$$V \longmapsto \{g_{ij} : E_{|U_{ij}} \simeq E_{|U_{ji}}\}$$

$$g_{ij} \in GL_n(\mathcal{O}^*_{U_{ij}})$$

<u>DEF</u> $\cdot\, \alpha \in H^2(X, \mathcal{O}_X^*)$ $\quad \alpha = \{\alpha_{i,j,k}\}$

UN FIBRATO TWISTATO ~~PER~~ DI VETTORI PER α È COLLEZIONE

$g_{i,j} \in GL_n(U_{i,j})$ | $- g_{i,i} = 1_n$

$- g_{i,j} = g_{i,j}^{-1}$

$- g_{i,j} \circ g_{j,k} \circ g_{k,i} = \alpha_{i,j,k} 1_n$

$\left\{ \begin{array}{l} \text{OPPURE COLLEZIONI DI FIBRATI } V_i \text{ SU } U_i \text{ + ISO} \\ \quad g_{i,j} : V_{i,j} \xrightarrow{\sim} V_{j,i} \quad | \quad \ldots \end{array} \right\}$

OSS SE CAMBIO α CON UN COBORDO DI α, LA DEFINIZIONE È EQUIVALENTE.

$\mathrm{VECT}(X,\alpha)$ o $\mathrm{COH}(X,\alpha)$ CATEGORIA DEGLI α-MODULI

OSS $V_1 \in \mathrm{VECT}(X,\alpha_1)$, $V_2 \in \mathrm{VECT}(X,\alpha_2)$ $M \in \mathrm{COH}(X)$

- $V_1^{-1} \in \mathrm{VECT}(X,\alpha_1^{-1})$,
- $V_1 \otimes V_2 \in \mathrm{VECT}(X,\alpha_1\alpha_2)$
- $\mathrm{HOM}(V_1,V_2) \in \mathrm{VECT}(X, \alpha_1^{-1}\alpha_2)$

- $f: Y \to X$ $M \in \mathrm{COH}(X,\alpha)$,
  - $f^* M \in \mathrm{VECT}(X, f^*\alpha)$
  - $f_* M$ "$\in$" $\mathrm{VECT}(X,\alpha)$

- $\mathrm{END}(V) \in \mathrm{VECT}(X)$

ES $\mathcal{L}$ FIBRATO IN RETTE. $\{g_{ij}\}$ $\quad \delta_n(\mathcal{L}) = \alpha_{ijk}$

SCEGLI RADICI n-ESIME h_{ij} ~~DA~~ DI g_{ij} $\quad \alpha^n = 1$ $\quad \{\alpha_{ijk}\}$

ALLORA $\{h_{ij}\} =: \mathcal{L}^{\frac{1}{n}}$ È UN FASCIO -TWISTATO PER α.

$$\left(\mathcal{L}^{\frac{1}{n}}\right)^n = \mathcal{L}$$

ES SE $\exists\, \mathcal{L}' \mid (\mathcal{L}')^n = \mathcal{L}$ $\quad \delta_n(\mathcal{L}) = 1$

QUINDI $\mathcal{L}^{\frac{1}{n}}$ È FASCIO. $\quad \mathcal{L}'$

ES $V^{EVECT(X,\alpha)}_{(\sim)}$ $\quad h_{ij}$ $\quad g_{ij} \circ g_{jk} \circ g_{ki} = \alpha_{ijk} 1_n$

ES V FIBRATO, $\mathcal{L} = DET(V)$ DI RANGO n

$$\delta_n(DET(V)) = \alpha$$

$V \otimes \mathcal{L}^{\frac{1}{n}}$ È FASCIO α-TWISTATO

~~DATI~~ CON $DET\left(V \otimes \mathcal{L}^{\frac{1}{n}}\right) = 1$

$\Lambda^n V \xrightarrow{\Lambda^n g_{ik}} \Lambda^n$

$v_1 \wedge \dots \wedge v_n \mapsto g_{ik}(v_1) \wedge g_{ik}(v_2) \dots$

$$\Lambda^n g_{ik} \circ g_{jk} \circ g_{ki}(v_1 \dots v_n) = \Lambda^n \alpha_{ijk} 1_k (v_1 \dots v_n) = \alpha_{ijk}^n$$

ES Supponiamo che $\alpha \in H^2(X, \mu_n)$ tale che $\alpha = \delta(\mathcal{L})$, $\mathcal{M}$ fibrato twistato per α. Allora $\mathcal{E} = \mathcal{M} \otimes \tilde{\mathcal{L}}^{-1}$ è fibrato vero

twisto, $\mathcal{L}$ per α, $\tilde{\mathcal{L}}$ e $\mathcal{M} = \mathcal{E} \otimes \tilde{\mathcal{L}}$

ATIYAH

CLASSI DI CHERN, ATIYAH E SEMIREGOLARITÀ.

$\mathcal{M} \in \mathrm{Vect}(X, \alpha) \qquad \alpha \in H^2(X, \mu_n)$

$$\begin{tikzcd} X \arrow[r] & \mathfrak{X} \arrow[r] \arrow[dr, bend right, "\pi_1"'] \arrow[rr, bend left, "\pi_2"] & X \times X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \\ & & X & \end{tikzcd}$$

$\pi_1^* \alpha_{|\mathfrak{X}} = \pi_2^* \alpha_{|\mathfrak{X}}$

$\pi_1^* \mathcal{M} \in \mathrm{Vect}(\mathfrak{X}, \pi_1^* \alpha)$
$= \mathrm{Vect}(\mathfrak{X}, \pi_2^* \alpha)$

$0 \to \iota_* \Omega^1_X \to \mathcal{O}_{\mathfrak{X}} \to \iota_* \mathcal{O}_X \to 0$

$0 \to \iota_* \Omega^1_X \otimes \pi_1^* \mathcal{M} \to \pi_1^* \mathcal{M} \to \iota_* \mathcal{O}_X \otimes \pi_1^* \mathcal{M} \to 0$

ESEMPI DI FASCI IN $VECT(\mathcal{X}, \pi_1' \alpha)$

$$0 \to \Pi_2 \left(\iota_0 \Omega^1_X \otimes \pi_1' M\right) \to \pi_{2,0}' \pi_1' M \to \pi_{2,0}\left(\iota_0 \mathcal{O}_X \otimes \pi_1' M\right) \to 0$$

IN $VECT(X, \alpha)$

$$0 \to M \otimes \Omega^1 \longrightarrow \,?\, \longrightarrow M \to 0 \qquad (*_M)$$

DEF $(*) \in Ext^1_\alpha(M, M \otimes \Omega^1) := Hom_{D^b(X,\alpha)}(M, M \otimes \Omega^1[1])$ $\qquad H^1(X, END(X))$

È LA CLASSE DI ATIYAH DI M $\qquad At_M$

$$(*)^n \in Ext^n(M, M \otimes \Omega^n) := Hom(M, M \otimes \Omega^n[n])$$

DEF $c_k(M) := \underline{Tr((*)^k)}$

$$Hom_{D^b(X,\alpha)}(M, M \otimes \Omega^1[1]) \simeq Hom_{D(X)}(\mathcal{O}_X, M \otimes M^\vee \otimes \Omega^1[1]) \simeq H^1(X, END(M) \otimes \Omega^1[1])$$

MAPPA DI SEMIREGOLARITÀ

$$\mathrm{Ext}^2(M,M) \xrightarrow{AT^q} \prod \mathrm{Ext}^{q+2}(M, M\otimes \Omega^q) \xrightarrow{TR} \prod H^{q+2}(\Omega^q)$$

TEOREMA (MARKMAN). SUPPONIAMO CHE

$Y \to X$ FAMIGLIA DI VARIETÀ ABELIANE. $\mathcal{E}_x / Y_x$

$\mathcal{E}_x \in \mathrm{EVOLT}(Y_x, \alpha)$, $\alpha \in H^2(X, \mathcal{M}_n)$ SEMIREGOLARE.

$ch_0(\mathcal{E}_x) \in H^0(\Pi_1(X))$. ALLORA $x \in U \subseteq X$ $\mathcal{E}|_{Y_U}$ ($\mathcal{E}_x = \mathcal{E}_x$

IN PARTICOLARE $\gamma(ch_0(\mathcal{E}_x))$ RIMANE ALGEBRICA $\forall \gamma$

LEMMA $V \in \mathrm{EVOLT}(X)$,

$V \otimes \det(V)^{\frac{1}{n}}$ È SEMI-REGOLARE $\Leftrightarrow$ V È SEMIREGOLARE

$$ch_0(\mathcal{E} \otimes \det^{\frac{1}{n}}) = ch_0(\mathcal{E})\, ch\left(\frac{1}{n} c_1(\det(\mathcal{E}))\right)$$

QUINDI LEMMA + TEOREMA $\Rightarrow$ TEO ?