OBSTRUCTION TO APPROXIMATING LOCAL SOLUTIONS COMING FROM 2-DIFFERENTIAL FORMS (JOINT WITH RACHEL NEWTON, MARGHERITA PAGANO)

1 WEAK APPROXIMATION

K NUMBER FIELD, Ω SET OF PLACES OF K $\left\{ \begin{array}{ll} \text{PRIMES OF } O_K & \text{FINITE} \\ K \hookrightarrow \mathbb{C} & \text{INFINITE.} \end{array} \right.$

$v \in \Omega$, $K_v = \left\{ \begin{array}{ll} \text{COMPLETION OF } K \text{ W.R.T. } v. & [K_v : \mathbb{Q}_p] \text{ FINITE} \\ \mathbb{R} & \text{IF } v : K \hookrightarrow \mathbb{R} \\ \mathbb{C} & \text{IF } v : K \hookrightarrow \mathbb{C} \end{array} \right.$

$X \subseteq \mathbb{P}^n_K$ DEFINED BY $\{ f_1, \ldots, f_r \in K[x_1 \ldots x_{n+1}] \}$ HOMOGENOUS

$X(K) := \{$ SOLUTIONS OF $f_1 = \ldots f_r = 0$ IN $K \}$

IMPOSSIBLE TO COMPUTE

$$X(K) \hookrightarrow \prod_{v \in \Omega} X(K_v) = X(\mathbb{A}_K)$$

$$x \longmapsto (x_v)_{v \in \Omega}$$

COUNTABLE

HUGE

ASSUME $X(\mathbb{A}_K) \neq \emptyset$

WHAT IS $X(K)$ INSIDE $X(\mathbb{A}_K) = ?$

$X(K_v)$ IS A TOPOLOGICAL SPACE INDUCED BY

$$X(K_v) \subseteq \frac{K_v^{n+1} \setminus \{0\}}{K_v^*} = \mathbb{P}^n_{K_v}$$

$\Rightarrow \prod X(K_v) = X(\mathbb{A}_K)$ IS A TOPOLOGICAL SPACE.

DEF X HAS WEAK APPROXIMATION (WA) OVER K

IF $\overline{X(K)} = X(\mathbb{A}_K)$.

RMK ① X HAS WA $\Leftrightarrow \forall S \subseteq \Omega_K$ $|S| < +\infty$

$X(K) \to \prod_{v \in S} X(K_v)$ DENSE

② IF $|S| < +\infty$ AND $\forall v \in S$ IS FINITE $(n_v) \in \mathbb{N}^{|S|}$

$X(K) \to \prod X(K_v)$ DENSE $\Leftrightarrow X(K) \to \prod X(\mathcal{O}_K / \mathfrak{p}_v^{n_v})$ SURJECTIVE.

EX $X \subseteq \mathbb{P}^2$ SMOOTH

• $X \subseteq \mathbb{P}^2$ $\deg(X) = 1$ $X \cong \mathbb{P}^1$ HAS WA

(CHINESE REMINDER THEOREM)

• $\deg(X) = 2$ X HAS WA (HASSE-MINKOWSKY)

• $\deg(X) \geq 3$ - X HAS NOT WEAK APPROXIMATION IF $X(\mathbb{A}_K) \neq \emptyset$.

(FALTINGS (HARARI 2000))

X HAS NOT WA OVER $\forall L \supseteq K$ BIG ENOUGH.

THM 1 (A.N.P.) $X \subseteq \mathbb{P}^3$ SMOOTH. SURFACE.

IF $\deg(X) \geq 4$ $\exists$ FINITE EXTENSION $K \subseteq L$ SUCH THAT $\forall L \subseteq F$, X HAS NO WA OVER F

RMK (1) IF $\deg(X) \leq 3$ THEN X HAS WA OVER $\forall$ BIG ENOUGH FINITE EXTENSION.

(2) $\deg(X) = 4$ DUE TO [illegible]-NEWTON. (K3 SURFACES)

X SMOOTH

$H^0(X, \Omega^n) = \{$ GLOBAL DIFFERENTIALS n-FORMS ON $X \}$

LOCALLY $\sum f_i \, dx_{i_1} \wedge \dots \wedge dx_{i_n}$ IF $x_1 \dots x_d$ ARE LOCAL COORDINATES ON X. THE GLUE TOGETHER.

EX $X: y^2 = x^3 - x \subseteq \mathbb{P}^2$ $2y\,dy = (3x^2 - 1)\,dx$

$\frac{dx}{y}$ IS A GLOBAL 1-FORM IF $y = 0$?

$$\frac{dx}{y} = \frac{2y\,dy}{(3x^2-1)y} = \frac{2\,dy}{3x^2-1}$$ WELL DEFINED WHEN $y = 0$

EX • $X \subseteq \mathbb{P}^2$ smooth $\quad H^0(X, \Omega^2) \neq 0 \iff \deg(X) \geq 3$

• $X \subseteq \mathbb{P}^3$ surface $\quad$ // $\quad H^0(X, \Omega^2) \neq 0 \iff \deg(X) \geq 4$

THM (An. P). X smooth. IF $H^0(X, \Omega^2) \neq 0$

$\exists\, E \supseteq K$ finite $\mid \forall\, F \supseteq L$ $\;X$ has no W.A. over F.

RMK Wittenberg conjecture:

IF X has WA over $\forall\, L \supseteq K$ THEN X is rationally connected

$\Rightarrow H^0(X, \Omega^i) = 0$ th

② OBSTRUCTIONS

EX (Lind-Reichardt '40) $X: 2y^2 - x^4 + 17z^4 = 0$, $X(\mathbb{A}_K) \neq \emptyset$ BUT $X(K) = \emptyset$

Why $X(K) = \emptyset$? Let $(x_0 : y_0 : z_0)$ solution with no common factors. IF $p \mid y_0$ THEN 17 is a square mod p $\underset{\text{quadratic reciprocity}}{\Longrightarrow}$ p is a square modulo 17.

SINCE $\underset{6\cdot 6}{\underset{\shortparallel}{2}}$ AND $\underset{4\cdot 4}{\underset{\shortparallel}{-1}}$ ARE SQUARES MOD 17 $\Rightarrow$ y_0 IS SQUARE IN $\mathbb{Q}_{17}$

$\Rightarrow$ $2 = \dfrac{x^4}{y^2}$ IS A FOURTH POWER IN $\mathbb{Q}_{17}$ $\lightning$.

WE SAY THAT THERE IS A <u>RECIPROCITY OBSTRUCTION</u>.

<u>THM</u> (MANIN '70) X/K SMOOTH

$\exists$ A TORSION GROUP $Br(X)$ AND A PAIRING

$$\langle , \rangle : X(\mathbb{A}_K) \times Br(X) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

SUCH THAT $X(K) \subseteq X(\mathbb{A}_K)^{Br(X)} := \{ (x_v) \mid \forall \alpha \in Br(X)\ \langle \alpha, (x_v) \rangle = 0 \}$

<u>EX</u> $\text{"}\underset{\alpha}{\underset{\in}{Br(X)[2]}}\text{"} = \langle \begin{smallmatrix} Y \\ \downarrow \\ X \end{smallmatrix} \mid \forall Q \in X \ Y_Q$ IS A SMOOTH CONIC $\rangle$?

$$inv_\alpha^{K_v} : X(K_v) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

$$Q \longmapsto \begin{cases} 0 & \text{SE } Y_Q(K_v) \neq \emptyset \\ 1 & \text{SE } Y_Q(K_v) = \emptyset . \end{cases}$$

$$X(\mathbb{A}_K) \times Br(X) \to \mathbb{Z}/\mathbb{Z}$$
$$((x_v), \alpha) \longmapsto \sum_v \mathrm{inv}_\alpha^{K_v}(x_v)$$

IF $(x_v) = (x) \in x \in X(K)$, THEN. $K_x \times_K K_v = K_{x_v}$

$$\sum_v \mathrm{inv}_\alpha^{K_v}(x_v) = 0 \quad \text{FROM THE PRODUCT FORMULA.}$$

EX IN LIND NOTCHES ONE CAN TAKE THE FAMILY $17a^2 + y_b b^2 + c^2 = 0$ OVER X TO SHOW THAT $X(\mathbb{A}_K)^{Br}(X) = \phi$. (ONE CAN EXAMINE WHERE $y = 0$)

IN GENERAL, GIVEN $\alpha \in Br(X)[n]$

$$\mathrm{inv}_\alpha^{K_v} : X(K_v) \to \mathbb{Z}/n\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$$

AND $(x_v, \alpha) := \sum_v \mathrm{inv}_\alpha^{K_v}(x_v)$

OBS ① $X(\mathbb{A}_K)^{Br} \subseteq X(\mathbb{A}_K)$ IS CLOSED.

$\Rightarrow$ IF $X(\mathbb{A}_K)^{Br} \neq X(\mathbb{A}_K)$, X HAS NOT WA.

② IF $\exists\, \alpha \in Br(X)$ AND $v \in \Omega_K$ |

$\mathrm{inv}^{\alpha}_{K_v}$ IS NOT CONSTANT, THEN

$$X(\mathbb{A}_K)^{Br} \subsetneq X(\mathbb{A}_K) \quad \left(\text{IF } X(\mathbb{A}_K) \neq \emptyset . \right)$$

<u>THM</u> 3 (ANP) X/K SMOOTH WITH $H^0(X,\Omega^2)$. $\forall$ PRIME $p \gg 0$,

$\exists$ A FINITE EXTENSION $L \supseteq K$, $\alpha \in Br(X_L)[p]$

SUCH THAT $\forall$ $F \supseteq L$ $\forall v \in \Omega_F$ $v | p$

$\mathrm{inv}^{\alpha}_{L_v}$ IS NOT CONSTANT. $\Rightarrow$ $X(\mathbb{A}_F)^{Br(X_F)} \subsetneq X(\mathbb{A}_F)$

<u>OBS</u> ① ESSENTIAL TO TAKE α OF p-TORSION FOR A PLACE OVER $v | p$. IF $v \nmid p$ THIS CANNOT HAPPEN. (SWINNERTON-DYER PENSAVA NON FOSSE POSSIBILE)

② $H^0(X,\Omega^2)$ IS A K-VECTOR SPACE (TORSION FREE.)

$$\begin{array}{c} H^0(X,\Omega^2) \\ \updownarrow \\ H^0(X_{\mathbb{F}_2},\Omega^2) \\ \updownarrow \text{ INTEGRAL } p\text{-ADIC HODGE THEORY} \\ Br(X) \end{array}$$

? $Br(X) \leftrightarrow H^0(X,\Omega^2)$

$Br(X)$ IS A TORSION GROUP

③ EXAMPLES

① A K3 (PAGANO 2023)

$$X \subseteq \mathbb{P}^5_{(x_1:x_2:x_3:y_1:y_2:y_3)}$$

$$x_1^2 + x_2x_3 + y_1^2 + x_1y_1 = 0$$
$$x_2^2 - x_1x_3 + y_2^2 + x_2y_2 = 0$$
$$x_3^2 - x_1x_2 + y_3^2 + x_3y_3 = 0$$

$\alpha : Y \to X$ DATA DA

$$\frac{x_2}{x_1}a^2 + \frac{x_3}{x_1}b^2 + c^2 = 0$$

$\in Br(X)[2]$

SI ESTENDE QUANDO $x_1, x_2, x_3 = 0$.

$ev_\alpha^{\mathbb{Q}_2} : \cancel{X}(\mathbb{Q}_2) \to \mathbb{Q}_2$ È NON COSTANTE.

COME TROVARLA? SIA $X_{\mathbb{F}_2}$ LA RIDUZIONE MODULO 2 DI X. È LISCIA.

$$H^0(X_{\mathbb{F}_2}, \Omega^2) = \left\langle d\log\left(\frac{x_2}{x_1}\right) \wedge d\log\left(\frac{x_3}{x_1}\right) \right\rangle$$

$$d\log(f) = \frac{df}{f}.$$

$$= \left(\frac{dx_2}{x_2} - \frac{dx_1}{x_1}\right) \wedge \left(\frac{dx_3}{x_3} - \frac{dx_1}{x_1}\right)$$

$$\frac{dx_2 \wedge dx_3}{x_2x_3} + \frac{dx_1 \wedge dx_3}{x_1x_3} + \frac{dx_1 \wedge dx_2}{x_1x_2}$$

② PRODOTTI DI CURVE ELLITTICHE.

E/K $\qquad E[p](\bar{K}) = (\mathbb{Z}/p)^2$

$\mu_p \subseteq K$

3150
2000
5150 $\simeq$ 4750

509

$$Br(X) \twoheadrightarrow \frac{End(E[p])}{End(E)/p}$$

$\tilde{f}$ $\qquad$ $\bar{\bar{f}}$

$\frac{End(E)}{p} \subseteq End(E[p])$

$End(E[p]) \simeq M_2(\mathbb{F}_p)$

E CON BUONA RIDUZIONE MODULO $v | p$,

$\mathcal{E}/\mathcal{O}_{K_v}$ MODELLO NERON, $\quad \mathcal{E}^{[p]}/\mathcal{O}_{K_v} := \ker(\mathcal{E} \xrightarrow{p} \mathcal{E})$

__THM__ (A.N.P). LET $f : E[p] \to E[p]$.

THEN $ev_{\tilde{f}}^{K_v}$ È COSTANTE $\iff$ f SI ESTENDE A $\mathcal{E}[p] \to \mathcal{E}[p]$

__EX__. IF $\mathcal{E}$ HAS ORDINARY REDUCTION. $\qquad$ CONNESSA $\quad$ È ÉTALE $\mathbb{Z}/p$

$$\mathcal{E}[p] \simeq \mu_p \times \mathbb{Z}/p \qquad End(\mathcal{E}[p]) = \cancel{\overset{0}{Hom(\mu_p, \mathbb{Z}/p)}} \oplus \overset{\mathbb{Z}/p}{Hom(\mu_p, \mu_p)}$$

$$(\mathbb{Z}/p)^2 \oplus \mathbb{Z}/p \qquad \simeq \overset{\mathbb{Z}/p}{Hom(\mathbb{Z}/p, \mathbb{Z}/p)} \oplus \overset{\mathbb{Z}/p \;\; 1 \mapsto \zeta_p}{Hom(\mathbb{Z}/p, \mu_p)}$$

- IF E HAS SUPERSINGULAR REDUCTION:

$E(p)$ IS NOT A PRODUCT $\Rightarrow$ $End(E[p])$ NON HA IDEMPOTENTI

$\Downarrow$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ NON SI SOLLEVA.

- IF E HAS CM COMPLEX CONJ $\sigma : E[p] \to E[p]$ NON SI SOLLEVA (SE $p > 2$) RK G_L