WILD BRAUER CLASSES VIA PRISMATIC COHOMOLOGY.

JOINT WORK WITH RACHEL NEWTON AND MARGHERITA PAGANO

1§ INTRODUCTION.

$[K:\mathbb{Q}_p] < +\infty$ $\quad$ k RESIDUE FIELD.

$\mathcal{O}_K$ RING OF INTEGERS.

X/K SMOOTH PROPER WITH GOOD REDUCTION.

$\mathfrak{X}/\mathcal{O}_K$ " MODEL

Y SPECIAL FIBER.

$$\alpha \in \mathrm{Br}(X) := H^2(X, \mathbb{G}_m) \quad K \subset L$$

$$ev_\alpha^L : X(L) \to \mathrm{Br}(L) \simeq \mathbb{Q}/\mathbb{Z}$$
$$P \longmapsto P^*\alpha$$

Q. WHEN IS ev_α^L NON CONSTANT??

(THEN α OBSTRUCTS LOCAL-TO-GLOBAL PRINCIPLE

VERY FEW EXAMPLES ARE KNOWN.

BLOCH-KATO + BRIGHT-NEWTON

- $H^0(Y, \Omega^2) = 0 \Rightarrow \exists\, L/K$ SUCH THAT $\forall\, F/L$ ev_α^F IS CONSTANT.

__THM 1__ (ABP)

ASSUME $H^0(Y, \Omega^2) \neq 0$ AND

(*) $\dim(H^i(Y, \Omega^j)) = \dim(H^i(X, \Omega^j))$. $\forall\, i, j$

THEN $\exists$ FINITE EXTENSION L/K AND $\alpha \in Br(X_L)[p]$ SUCH THAT

ev_α^F IS NOT CONSTANT $\forall\, F/L$

__RMK__ IF X IS DEFINED OVER A # FIELD M, X_{M_v} SATISFIES (*) $\forall$ BUT FINITELY MANY PLACES v

COR M NUMBERFIELD X/M SUCH THAT

$$X(N) \subseteq X(\mathbb{A}_N) \quad \forall\, N/M.$$

(↑ DENSE)

THEN $H^0(X, \Omega^2) = 0$.

RMK ① IF $\alpha \in \mathrm{Br}(X_L)[\ell]$ $\ell \neq p$.
$\exists\, L/K$ SUCH THAT ev_α^F IS CONST $\forall\, F/K$

② THM 1 KNOWN IF X ORDINARY (BRIGHT-NEWTON)

③ PROOF NOT CONSTRUCTIVE BUT SOMETIMES WE CAN GIVE MORE EXPLICIT RESULTS

EX $X = E \times E$ E ELLIPTIC CURVE

$$\mathrm{Br}(X)[p] = \frac{\mathrm{End}(E[p])}{\mathrm{End}(E)} \ni \alpha$$

PROOF: THM + PRISMATIC DIEUDONNÉ THEORY $\Rightarrow$ ev_α CONSTANT $(\Leftarrow)$ α EXTENDS TO $\mathcal{E}[p] \to \mathcal{E}[p]$

② PRISMATIC COHOMOLOGY (MOD p)

ASSUME $\hat{*}$ HOLDS (FOR ALL TATE NICE)

$\mathcal{O}_{\mathbb{C}_p^\flat} := \overline{k}[[t]]$ $\left(T_{\mathcal{O}_{\mathbb{C}_p}}^{\mathrm{af}}\Gamma\right)$ $\quad \mathcal{O}_{\mathbb{C}_p^\flat} \to \overline{k}$, $T \mapsto 0$

BHATT-SCHOLZE:

$\exists$ A COHOMOLOGY THEORY $H^n_\Delta := H^n_\Delta(X)$ (PRISM)

SUCH THAT:

- H^n_Δ IS FREE $\mathcal{O}_{\mathbb{C}_p^\flat}$-MODULE OF FINITE RANK $\left(= \dim\left(H^n_{ET}(X_{\overline{K}}, \mathbb{Z}/p)\right)\right)$
- $\varphi: H^n_\Delta \to H^n_\Delta$ φ-LINEAR WHICH IS ISO INVERTING T
- $H^n_\Delta[\frac{1}{T}] \otimes_{\mathcal{O}_{\mathbb{C}_p^\flat}} \overline{k} = H^n_{DR}(X_{\overline{k}})$ DE-RHAM COMPARISON
- $H^n_\Delta[\frac{1}{T}] \simeq H^n_{ET}(X_{\overline{K}}, \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{F}_p} \mathcal{O}_{\mathbb{C}_p^\flat}[\frac{1}{T}]$ ETALE COMPARISON

NOT $\alpha \in \mathbb{Q}$ $\quad \mathcal{O}(-\alpha) = \mathcal{O}_{\mathbb{C}_p^\flat} e \quad \varphi(e) = \alpha^d e$

EX

① $X = \mathbb{P}^1_{\mathbb{Z}/p}$

$$H^0_\Delta = \mathcal{O}(0)$$
$$H^1_\Delta = 0$$
$$H^2_\Delta = \mathcal{O}(-1)$$

② X ELLIPTIC CURVE

$H^0_\Delta = \mathcal{O}, \; H^2_\Delta = \mathcal{O}(-1)$

$$H^1_\Delta = \begin{cases} \mathcal{O} \oplus \mathcal{O}(-1) & \text{IF } X \text{ ORDINARY} \\ 0 \to \mathcal{O}(-\alpha) \to H^1_\Delta \to \mathcal{O}(1-\alpha) \to 0 \end{cases}$$

$\alpha \in \mathbb{Q}$ NOT SPLIT X SUPERSINGULAR

α APPEARING IN THE DIEU-TATE CLASSIFICATION

$$Br(X_{\bar{K}})[p]^{GB} \qquad (H^2_\Delta)^{\delta=T} = \{x \in H^2_\Delta \mid \delta(x) = Tx\}$$

$$\| $$

$$\{\alpha \in Br(X_{\bar{K}}) \mid \forall L \mid \alpha \in Br(X_L) \; \exists F/L \mid e_{v_F} \text{ CONSTANT}\}$$

<u>THM 2</u> (ANP) ← SYNTOMIC INTERPRETATION OF $Br(X_{\bar{K}})[p]^{GB}$, BMS II, PREVIOUS WORK OF BN.

ASSUME (*)

$$\operatorname{DIM}\left(\frac{Br(X_{\bar{K}})[p]}{Br(X_{\bar{K}})[p]^{GB}}\right) = \operatorname{RK}(H^2_\Delta(x)) - \operatorname{DIM}_{\mathbb{F}_p}\left(H^2_\Delta(x)^{\delta=T}\right)$$

↖ ANALOGUE OF $0 \to H^2_{\text{FPPF}} \to H^2_{\text{CRYS}} \xrightarrow{1-\delta} H^2_{\text{CRYS}} \to 0$

<u>THM 3</u> (ANP)

$$\text{IF } \operatorname{DIM}(H^2_\Delta(x)^{\delta=T}) = \operatorname{RK}(H^2_\Delta(x))$$

$$\Rightarrow H^0(X, W^2) = 0$$

<u>THM 3</u> + <u>THM 2</u> ⇒ <u>THM 1</u>

(3) NEWTON ABOVE HODGE (PROOF OF THM 3)

ACTION OF φ $\overset{?}{\longleftrightarrow}$ DIFFERENTIAL FORMS

NYGAARD IF $H^0(X,\Omega^2) \neq 0 \Rightarrow \varphi : H^2_{DR} \to H^2_{DR} \neq 0$

$\Rightarrow$ $\varphi : H^2_{\Delta} \to H^2_{\Delta}$ NOT DIVISIBLE BY d

($\uparrow$ DE RHAM) NOT ENOUGH!

EX IF $\mathcal{O}(-\alpha)^{\varphi = T} = \begin{cases} 0 & \alpha > 1 \\ \mathbb{F}_p & \alpha \leq 1 \end{cases}$

BUT $\varphi : \mathcal{O}(-\alpha) \to \mathcal{O}(-\alpha)$ DIVISIBLE BY $T \Leftrightarrow \alpha \geq 1$

SO WE WANT TO EXCLUDE STUFF LIKE: $H^2_{\Delta} = \mathcal{O}(-\alpha)$ FOR $0 \leq \alpha < 1$

LEMMA. $\exists$ FILTRATION F. $F^n \subseteq \ldots \subseteq F^0 = H^n_{\Delta}$

$$\frac{F^i}{F^{i+1}} \simeq \mathcal{O}(-\alpha_i) \qquad \alpha_i \in \mathbb{Q}$$

AND $\sum \alpha_i$ DOES NOT DEPENDS ON THE CHOICE OF $F^\bullet$

$TS(H^n_{\Delta}) := \sum \alpha_i$.

THM 4 (AMP) ASSUME (*)

$$TS(H^n_\Delta) = \sum_{i=0}^{n} i \, \dim(H^i(X, \Omega^{n-i}))$$

COR $TS(H^2_\Delta) = \mathrm{RANG}(H^2_\Delta) = r$

THM 4 $\Rightarrow$ **THM 3**

ASSUME $\dim(H^2_\Delta)^{S=T} = \mathrm{RANG}(H^2_\Delta) = r$

$$\dim\left((H^2_\Delta)^{S=T}\right) \leq \sum_{i=1}^{r} \dim\left(\left(\frac{F^i}{F^{i+1}}\right)^{S=T}\right) =$$

$$= \sum_{i=1}^{r} \dim\left(\mathcal{O}(-\alpha_i)^{S=T}\right) = r$$

$\Rightarrow \forall i \quad \alpha_i \geq 1$

THM 4 $\Rightarrow \sum^{r} \alpha_i = \mathrm{RANG} = r \Rightarrow \alpha_i = 1 \ \forall i$

H^2_J IS AN ITERATED EXTENSION OF $\mathcal{O}(-1)$

AND $\dim\left(\left(H^2_J\right)^{\varphi=T}\right) = R$

$\Downarrow$ ← SEMI-LINEAR ALGEBRA.

$H^2_J \simeq \oplus \mathcal{O}(-1)$

$\Downarrow$ ← DE RHAM COMPARISON.

φ ON $H^2_{DR} = H^2_J \otimes k$ IS TRIVIAL

NYGAARD

$\Rightarrow$ $H^0(\mathcal{X}, \Omega^2) = 0$ ∎